MATRIX GROUPS / I

Review of groups. Read appendix and do problems

Review linear algebra.

vector spaces, linear transformations

inner product

MAIN EXAMPLE OF A LIE GROUP

$$GL(n,\mathbb{R}) := \{ g \ n\times n \text{ matrix}, \ g \text{ invertible} \}$$
$$= \{ g \ n\times n \text{ matrix}, \ \det g \neq 0 \}.$$

$GL(n,\mathbb{R}) \subset M(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$

is an open set.

RIGID MOTIONS

$A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$|Av - Aw| = |v-w|$ for all $v, w \in \mathbb{R}^n$

Then $|Av| = |v|$ for all v,

NOTE: $|Av| = |v|$ not the same as $|Av - Aw| = |v-w|$

Examples: (1) translations

$T_v(x) = v + x$.

If $A(0) = v \neq 0$ $\quad T_{-v} \circ A(0) = T_{-v}(v) = v - v = 0$

satisfies $A(0) = 0$.

<u>Theorem</u>: $\{A : \mathbb{R}^n \to \mathbb{R}^n,\ |Av - Aw| = |v - w|$
$A(0) = 0\}$

forms a group.

We show that in fact more is true;

A is an invertible linear transformation

1.) We show $\langle Av_1, Av_2 \rangle = \langle v_1, v_2 \rangle$

$|v_2 - v_1|^2 = |v_2|^2 + |v_1|^2 - 2\langle v_2, v_1 \rangle$

so $\langle v_2, v_1 \rangle = \dfrac{|v_2|^2 + |v_1|^2 - |v_2 - v_2|^2}{2}$

$$\langle Av_2, Av_1 \rangle = \frac{|Av_2|^2 + |Av_1|^2 - |Av_2 - Av_1|^2}{2}$$

$$= \frac{|v_2|^2 + |v_1|^2 - |v_2 - v_1|^2}{2}$$

2.) $A(v_1 + v_2) = Av_1 + Av_2$, $\quad A(v_1 - v_2) = Av_1 - Av_2$

Compute $|A(v_1+v_2) - Av_1 - Av_2|^2 =$

$= |A(v_1+v_2)|^2 + |Av_1|^2 + |Av_2|^2 - 2\langle A(v_1+v_2), Av_1 \rangle - 2\langle A(v_1+v_2), Av_2 \rangle$

$+ 2\langle Av_1, Av_2 \rangle = |v_1+v_2|^2 + |v_1|^2 + |v_2|^2 - 2\langle v_1+v_2, v_1 \rangle$

$- 2\langle v_1+v_2, v_2 \rangle + 2\langle v_1, v_2 \rangle = |(v_1+v_2) - v_1 - v_2|^2 = 0$

same for second relation

3.) $A(cv) = cAv$

$$A(-v) = A(0-v) = A(0) - A(v) = -A(v)$$

$$A(mv) = mAv \qquad A(\tfrac{1}{n}v) = \tfrac{1}{n}Av$$

so $A(rv) = rAv$ for any $r \in \mathbb{Q}$

Now let $c \in \mathbb{R}$, $r_i \longrightarrow c$ $\quad r_i \in \mathbb{Q}$

$$|A(cv) - A(r_iv)| = |cv - r_iv| = |c - r_i| \cdot |v| \longrightarrow 0$$

$$|A(cv) \overset{\shortparallel}{-} r_iAv| \longrightarrow |A(cv) - cAv| = 0$$

So A is linear. Let $e_1, \ldots, e_n$ be an orthonormal basis.

$Ae_1, \ldots, Ae_n$ is also an o.n. basis

$$A(x_1e_1 + \cdots + x_ne_n) = x_1Ae_1 + \cdots + x_nAe_n$$

$$Ae_j = (a_{ij}) \qquad A \longleftrightarrow (a_{ij})$$

columns are orthonormal vectors

4.) Conversely, suppose A (is linear and) satisfies $\langle Av_1, Av_2 \rangle = \langle v_1, v_2 \rangle$. Then

$$|Av_1 - Av_2|^2 = \langle Av_1, Av_1 \rangle + \langle Av_2, Av_2 \rangle - 2\langle Av_1, Av_2 \rangle$$

$$= \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle - 2\langle v_1, v_2 \rangle = |v_1 - v_2|^2$$

1/25/4

5) $\langle Av_1, Av_2 \rangle = \langle A^T \cdot A v_1, v_2 \rangle = \langle v_1, v_2 \rangle$

is equivalent to $A^T \cdot A =$ Identity

So $A^{-1} = A^T$

<u>Theorem</u> $O(n) := \{A : A^T \cdot A = I\}$

is a group under matrix multiplication

Pf: $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1} = B^T \cdot A^T = (AB)^T$

<u>Corollary</u>: $O(n)$ is compact

<u>Pf</u>: Since the column vectors ~~are~~ form an o.n. basis, the set $O(n)$ is bounded.

If $A_i \to A$ & $A_i^T \cdot A_i = I$, ${}^t A \cdot A = I$

so $O(n)$ is also closed.

<u>NOTE</u>: $O(n)$ is closed under transpose, i.e. if $A \in O(n)$ then $A^T \in O(n)$.

$A^T \cdot A = Id \Rightarrow A \cdot {}^tA^T = Id \Leftrightarrow (A^T)^T \cdot A^T = Id$

<u>$SO(n)$</u> $:= \{ g \in O(n) : \det g = 1 \}$ is a closely related group.

<u>Remark</u>: $g \in O(n) \Leftrightarrow {}^t g \cdot g = I \Rightarrow \det {}^t g \cdot \det g = 1$

$\Leftrightarrow (\det g)^2 = 1 \Longleftrightarrow \det g = \pm 1$.

$\underline{O(2)}$; $\{e_1, e_2\} \longrightarrow \{Ae_1, Ae_2\}$

$Ae_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$; any unit vector can be written this way.

Then Ae_2 is a unit vector $\perp Ae_1$. Must be

$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$O(2) = SO(2) \cup SO(2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\underline{O(n)}$: $SO(n) \cup SO(n) \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$

$SO(3)$ Ae_1, Ae_2, Ae_3

Ae_1 is a unit vector. $\begin{pmatrix} \cos\theta\cos\varphi \\ \sin\theta\cos\varphi \\ \sin\varphi \end{pmatrix}$

(Spherical coordinates)

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So $Ae_1 = r_{12}(\theta)\, r_{13}(\varphi)\, e_1$, $r_{13}(-\varphi)\, r_{12}(-\theta) A e_1 = e_1$

so $r_{13}(-\varphi) r_{12}(-\theta) A$ stabilizes $x_2 e_2 + x_3 e_3$, and is in $SO(2)$.

$r_{23}(-\tau)\, r_{13}(-\varphi)\, r_{12}(-\theta) A = I$

$A = r_{12}(\theta)\, r_{13}(\varphi)\, r_{23}(\tau)$.

(θ, φ, τ) are coordinates for $SO(3)$.

called Euler angles. $0 \le \theta, \varphi, \tau < 2\pi$

$\therefore$ $SO(3)$ is pathwise connected.

DEFINITION: A matrix group is any closed subgroup of $GL(n, \mathbb{R})$.

E.G. $SO(n) \subset GL(n, \mathbb{R})$

but also $GL(n, \mathbb{R})_+ := \{g \in GL(n, \mathbb{R}) : \det g > 0\}$

On the other hand

$$(e^{i\theta}, e^{2\pi i\theta}) \qquad \theta \in \mathbb{R}$$

is a group, but not closed. So not a matrix group.

Embed $GL(n, \mathbb{R})$ in $GL(n+1, \mathbb{R})$ via

$g \mapsto \operatorname{diag}(g, \det g)$

The image is closed even in $M(n, \mathbb{R})$

Exponential Map:

$X \in M(n,\mathbb{R}) \longmapsto \exp(X) = I + \frac{X}{1!} + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + \cdots$

Proposition: The series converges for all X.

Pf: $|X \cdot Y| \leq |X| \cdot |Y|$ and $|X+Y| < |X| + |Y|$
where $|X| = \left(\sum x_{ij}^2\right)^{1/2}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \vec{a_1}\cdot\vec{b_1} & \vec{a_1}\cdot\vec{b_2} \\ \vec{a_2}\cdot\vec{b_1} & \vec{a_2}\cdot b_2 \end{pmatrix}$$

$(\vec{a_1}\cdot\vec{b_1})^2 \leq |a_1|^2 \cdot |b_1|^2$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a series with radius of convergence $R > 0$

Then $f(X) = \sum_{n=0}^{\infty} a_n X^n$ converges for any $|X| < R$

Proposition: Suppose $X \cdot Y = Y \cdot X$, Then
$\exp(X+Y) = \exp X \cdot \exp Y$

Bilinear Forms $\quad V$ a vector space

$(\, , \,) : V \times V \longrightarrow \mathbb{R}$

$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w)$

$(v, aw_1 + bw_2) = a(v, w_1) + b(v, w_2)$

Choose a basis $e_1, e_2, \ldots, e_n$

$v = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$w = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n$

(*) $(v, w) = \sum x_i y_j \, (e_i, e_j)$

Example: $V = \mathbb{R}^n$ with usual basis.

$\langle \, , \, \rangle = \sum x_i y_j$

is such a form.

Symmetric $\quad \langle v, w \rangle = \langle w, v \rangle$

Positive definite $\quad \langle v, v \rangle \geq 0 \quad = 0 \iff v = 0$

Nondegenerate $\quad (v, w) = 0 \;\; \forall w \implies v = 0$

e.g. $V = \mathbb{R}^2 \quad (v, w) = x_1 y_1 - x_2 y_2$

If $\; x_1 y_1 - x_2 y_2 = 0 \;\; \forall (y_1, y_2)$, then $x_1 = x_2 = 0$

1/27/2

Go back to (*)

Let $\langle\ ,\ \rangle$ be $\sum x_i y_i$ w.r.to to chosen basis. Let $a_{ij} := (e_i, e_j)$

$A := (a_{ij})$

$$(v,w) = \sum_j y_j \sum_i x_i a_{ij} = \sum_i x_i \sum_j a_{ij} y_j$$

$$(a_{ij}) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$= \langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, {}^T\!A\vec{y} \rangle$$

Change of basis M = change of basis matrix

A goes to ${}^T\!M \cdot A \cdot M$

(exercise)

Symmetric gives ${}^T\!A = A$

Nondegenerate gives A invertible

Classification up to $A \sim {}^T\!M \cdot A \cdot M$.

Assume A is symmetric

Quadratic Forms

$$\langle A\vec{x}, \vec{y} \rangle \longleftrightarrow Q(\vec{x}) := \langle A\vec{x}, \vec{x} \rangle$$

$$\sum a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j$$

$Q(x+y) - Q(x) - Q(y)$

$\langle Ax + Ay, Ax + Ay\rangle - \langle Ax, Ax\rangle - \langle Ay, Ay\rangle$

$= 2\langle Ax, y\rangle$.

Complete the square:

Suppose one of the $a_{ii} \neq 0$. By interchanging the order of the e_i, we may assume $a_{11} \neq 0$.

$$a_{11}\left(x_1^2 + 2\frac{a_{12}}{a_{11}}x_1x_2 + \cdots + 2\frac{a_{1n}}{a_{11}}x_1x_n\right)$$

$$+ (\text{terms in } x_2, \ldots, x_n)$$

$$= a_{11}\cdot\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \cdots + \frac{a_{1n}}{a_{11}}x_n\right)^2$$

$$+ (\text{terms in } x_2, \ldots, x_n)$$

Change variables $x_1' = x_1 + \frac{a_{12}}{a_{11}}x_2 + \cdots + \frac{a_{1n}}{a_{11}}x_n$

$$x_2' = x_2$$
$$\vdots$$
$$x_n' = x_n$$

$$Q(x') = a_{11}x_1'^2 + Q_1(x_2', \ldots, x_n')$$

Continue the process

Continuing this way, we can rewrite Q

as $a_1 z_1^2 + \cdots + a_n z_n^2$.

$a_1, \ldots, a_n \in \mathbb{R} \longleftrightarrow A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$.

Special Case: Suppose all $a_{ii} = 0$.

Some term has to be nonzero; say $a_{12} \neq 0$

change variables $x_1 = x_1' - x_2'$, $x_2 = x_1' + x_2'$

so $2a_{12}(x_1'^2 - x_2'^2)$ + other terms

Sesquilinear Forms

V a complex space.

$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$

(a) $\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha\langle \vec{x}, \vec{z} \rangle + \beta\langle \vec{y}, \vec{z} \rangle$

(b) $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

Main Example:

$\langle \vec{x}, \vec{y} \rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$

$\langle \vec{x}, \vec{x} \rangle \geq 0$ & $= 0 \iff \vec{x} = 0$.

$U(n) = \{ g \in GL(n, \mathbb{C}) \mid \langle g\vec{x}, g\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

$\iff \langle g\vec{x}, g\vec{x} \rangle = \langle \vec{x}, \vec{x} \rangle$

$\iff A^{-1} = {}^T\overline{A}$ $\quad {}^T\overline{A} \cdot A = Id$

$SU(n) = \{ g \in U(n) \mid \det g = 1 \}$.

In general, if $g \in U(n)$,

$${}^t\bar{g} \cdot g = I \Rightarrow \overline{\det g} \cdot \det g = 1$$

$\Leftrightarrow |\det g| = 1$. all numbers of the form $e^{i\theta}$, $\theta \in \mathbb{R}$

Structure of $SU(2)$.

$$\{e_1, e_2\} \longmapsto \{g \cdot e_1, g \cdot e_2\}$$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow g \cdot e_1 = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \quad g \cdot e_2 = \begin{pmatrix} \beta \\ \delta \end{pmatrix}$$

unit vectors

$$\begin{cases} |\alpha|^2 + |\gamma|^2 = 1 , & |\beta|^2 + |\delta|^2 = 1 \\ \alpha\bar{\beta} + \gamma\bar{\delta} = 0 \end{cases}$$

$$\alpha\delta - \beta\gamma = 1$$

$$\alpha = r_1 e^{i\theta_1} \qquad \gamma = r_2 e^{i\theta_2}$$

with $r_1^2 + r_2^2 = 1$ so

$$\alpha = \cos\theta \, e^{i\theta_1} \qquad \gamma = \sin\theta \, e^{i\theta_2}$$

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \, e^{i\theta_1} \\ \sin\theta \, e^{i\theta_2} \end{pmatrix}$$

$$= g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{-i\theta_2} \end{pmatrix} g}_{g'} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$g'\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{unit vector} \perp \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$0 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix} \right\rangle = \beta$$

So $g' \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tau} \end{pmatrix}$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tau} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}$$

If $\det g = 1$,

$$g = \begin{pmatrix} e^{i\tau_1} & 0 \\ 0 & e^{-i\tau_1} \end{pmatrix}\begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} e^{i\tau_2} & 0 \\ 0 & e^{-i\tau_2} \end{pmatrix}$$

3 parameters. <u>NOTE</u>: $SU(2)$ is a <u>real</u> group

Group Homomorphisms

$\pi : G \longrightarrow H$ $\qquad \pi(g_1 g_2) = \pi(g_1)\pi(g_2)$

$\pi(e) = e$.

$G = SU(2)$ vs $H = SO(3)$

2/1/3

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \right\} \quad a,b,c \in \mathbb{R}$$

$$\langle Xv, w \rangle + \langle v, Xw \rangle = 0$$

$$^t\overline{X} = -X$$

If $g^{-1} = {}^t\overline{g}$, $^t\overline{X} = -X$ then

$$^t\overline{(gXg^{-1})} = -gXg^{-1}.$$

So we get a map

$$G = SU(2) \longrightarrow GL(3)$$

In fact this presentation of $\mathbb{R}^3$ has an inner product which is naturally preserved by the action we defined:

$$\langle X, Y \rangle := tr({}^t\overline{X} Y)$$

$$\begin{pmatrix} ia_1 & b_1+ic_1 \\ -b_1+ic_1 & -ia_1 \end{pmatrix}, \begin{pmatrix} -ia_2 & -b_2-ic_2 \\ b_2-ic_2 & +ia_2 \end{pmatrix} =$$

$$= \begin{pmatrix} +a_1a_2 + b_1b_2 + c_1c_2, & * \\ * & a_1a_2 + b_1b_2 + c_1c_2 \end{pmatrix} =$$

$$= 2(a_1a_2 + b_1b_2 + c_1c_2$$

$$\mathrm{tr}\left(g\overset{T}{\bar{X}}g^{-1}\cdot gYg^{-1}\right)=\mathrm{tr}\left(g\overset{T}{\bar{X}}Yg^{-1}\right)=\mathrm{tr}\left(g^{-1}g\overset{T}{\bar{X}}Y\right)$$
$$=\mathrm{tr}\left(\overset{T}{\bar{X}}Y\right).$$

Properties of trace:

$\mathrm{tr}(AB)=\mathrm{tr}(BA)$

$\mathrm{tr}(A+B)=\mathrm{tr}A+\mathrm{tr}B$

$\mathrm{tr}(cA)=c\,\mathrm{tr}A.$

So the map

$$SU(2)\longrightarrow GL(3)$$

actually has image in $O(3)$.

In fact the image is in $SO(3)$, determinant is 1.

Example of a homomorphism

G a group

$\mathrm{Aut}(G):=\{\varphi:G\to G \mid \varphi(g_1g_2)=\varphi(g_1)\varphi(g_2),\ \varphi(e)=e\}$

This forms a group.

$\mathrm{Adj}:G\longrightarrow \mathrm{Aut}(G)\qquad \mathrm{Adj}(g)(h):=ghg^{-1}$

Semidirect Product

G, H groups, $\Psi : G \longrightarrow Aut(H)$ a group homomorphism.

$G \ltimes H := G \times H$ but multiplication is given by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, \Psi(g_2) h_1 \cdot h_2)$$

Examples:

(1) $G = (O(3), \cdot)$ $V \simeq (\mathbb{R}^3, +)$

$\Psi : G \longrightarrow Aut(V)$ $g \mapsto \Psi(g) v = g \cdot v$

$G \ltimes V$ is the ~~aa~~ group of rigid motions.

$SO(3) \ltimes V$ is the group of "orientation preserving" rigid motions.

(2) Upper triangular group.

$G : \left\{ \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} : a_1 \cdot a_2 \neq 0 \right\}$.

$G = A \ltimes \mathbb{R}$ $A : \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1 \cdot a_2 \neq 0 \right\}$

$\mathbb{R} : \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ with multiplication

$\Psi : A \longrightarrow Aut(\mathbb{R})$ $\Psi(a) v = a v a^{-1}$

Inner Product on the vector space of $n \times n$ matrices, real or complex.

$V = M_{m\times n} = \{ n \times m$ matrices with coefficients in $F = \mathbb{R}$ or $\mathbb{C} \}$

$X = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

m rows $\quad n$ columns

${}^T X = (a_{ji}) \qquad n \times m$ matrix

$\overline{X} = (\overline{a_{ij}}) \qquad m \times n$ matrix

$\langle \, , \rangle : V \times V \longrightarrow \mathbb{C} \; (\mathbb{R})$.

$\langle X, Y \rangle = tr(X \cdot {}^T\overline{Y})$

(1) linear in $\langle \cdot , \text{fixed} \rangle$

skew linear in $\langle \text{fixed}, \cdot \rangle$

(2) $\langle X, Y \rangle = \overline{\langle Y, X \rangle}$

Follows from the fact that

$\langle X, Y \rangle = \sum_{i,j} a_{ij} \overline{b_{ij}}$

$GL(m,\mathbb{R}) \times GL(n,\mathbb{R})$ <u>acts</u> on $M_{m\times n}$

$(g_1, g_2)\cdot X = g_1 X \cdot g_2^{-1}$

<u>Proposition:</u> If $g_1^{-1} = {}^T\overline{g_1}$, $g_2^{-1} = {}^T\overline{g_2}$, then

$\langle gX, gY\rangle = \langle X, Y\rangle$

<u>Proof:</u> $(g_1 X g_2^{-1})\cdot {}^T\overline{(g_1 Y g_2^{-1})} =$

$= g_1 X g_2^{-1} \cdot {}^T\overline{(g_2^{-1})}\cdot {}^T\overline{Y}\cdot {}^T\overline{g_1} = g_1 X \cdot {}^T\overline{Y} g_1^{-1}$

Use properties of trace.

<u>REMARK</u>: (1) $(g_1, g_2) \longmapsto \varphi_{(g_1,g_2)} \in Aut(M_{m\times n})$

is a group homomorphism

(2) $GL(m,F)\times GL(n,F) \longrightarrow GL(mn, F)$.

The case $\Phi: SU(2) \longrightarrow SO(3)$

is a bit different; same flavor though.

(3) the proposition says that $g \mapsto \varphi_g$ takes

$O(m)\times O(n) \longrightarrow O(mn)$.

$V = \mathbb{C}^2$

Define $\langle (z_1, z_2), (w_1, w_2) \rangle = z_1 w_1 + z_2 w_2$

What is $SO(\langle \, , \, \rangle)$?

The basis e_1, e_2 satisfies

$(e_1, e_1) = (e_2, e_2) = 1, \quad (e_1, e_2) = 0.$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1$$

$$\alpha^2 + \gamma^2 = 1 \qquad \alpha\beta + \gamma\delta = 0$$

$$\beta^2 + \delta^2 = 1$$

$$\begin{pmatrix} \alpha & -t\gamma \\ \gamma & t\alpha \end{pmatrix} \text{ for some } t; \quad t(\alpha^2 + \gamma^2) = 1 \Rightarrow t = 1$$

$$\begin{pmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{pmatrix} \quad \text{with} \quad \alpha^2 + \gamma^2 = 1$$

$$\alpha = r_1 e^{i\theta_1} \qquad \gamma = r_2 e^{i\theta_2}$$

$$\alpha = \cos z \qquad \beta = \sin z$$

with z complex.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$$

Image of $SU(2)$ in $O(3)$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} ia' & b'+ic' \\ -b'+ic' & -ia' \end{pmatrix}$$

$$(a, b, c) \longmapsto (a, \cos 2\theta\, b + \sin 2\theta\, c, \cos 2\theta\, b - \sin 2\theta\, c)$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

sends

$$(a,b,c) \longmapsto (\cos 2\theta\, a, b, a\sin 2\theta - c\cos 2\theta)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{pmatrix}, \quad \begin{pmatrix} \cos 2\theta & 0 & \sin 2\theta \\ 0 & 1 & 0 \\ -\sin 2\theta & 0 & \cos 2\theta \end{pmatrix}$$

PROPOSITION: Any $g \in SO(3)$ can be written as

$$\begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

(Exercise)

Bilinear Forms $\quad V$ a vector space

$(\, , \,) : V \times V \longrightarrow \mathbb{R}$

$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w)$

$(v, aw_1 + bw_2) = a(v, w_1) + b(v, w_2)$

Choose a basis $e_1, e_2, \dots, e_n$

$v = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$w = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n$

(*) $(v, w) = \sum x_i y_j (e_i, e_j)$

Example: $V = \mathbb{R}^n$ with usual basis.

$\langle \, , \, \rangle = \sum x_i y_j$

is such a form.

Symmetric $\quad \langle v, w \rangle = \langle w, v \rangle$

Positive definite $\quad \langle v, v \rangle \geq 0 \quad = 0 \iff v = 0$

Nondegenerate $\quad (v, w) = 0 \ \forall w \implies v = 0$

e.g. $V = \mathbb{R}^2 \quad (v, w) = x_1 y_1 - x_2 y_2$

If $\quad x_1 y_1 - x_2 y_2 = 0 \ \forall (y_1, y_2)$, then $x_1 = x_2 = 0$

Go back to (*)

Let $\langle\, , \rangle$ be $\sum x_i y_i$ w.r.to to chosen basis. Let $a_{ij} := (e_i, e_j)$

$A := (a_{ij})$

$$(v, w) = \sum_j y_j \sum_i x_i a_{ij} = \sum_i x_i \sum_j a_{ij} y_j$$

$$(a_{ij}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, {}^T\!A\vec{y} \rangle$$

Change of basis M = change of basis matrix

A goes to ${}^T M \cdot A \cdot M$

(exercise)

Symmetric gives ${}^T A = A$

Nondegenerate gives A invertible

Classification up to $A \sim {}^T M \cdot A \cdot M$.

Assume A is symmetric

Quadratic Forms

$$\langle A\vec{x}, \vec{y} \rangle \longleftrightarrow Q(\vec{x}) := \langle A\vec{x}, \vec{x} \rangle$$

$$\sum a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j$$

$Q(x+y) - Q(x) - Q(y)$

$\langle Ax + Ay, Ax + Ay \rangle - \langle Ax, Ax \rangle - \langle Ay, Ay \rangle$

$= 2 \langle Ax, y \rangle$.

<u>Complete the square:</u>

Suppose one of the $a_{ii} \neq 0$. By interchanging the order of the e_i, we may assume $a_{11} \neq 0$.

$$a_{11}\left(x_1^2 + 2\frac{a_{12}}{a_{11}}x_1x_2 + \cdots + 2\frac{a_{1n}}{a_{11}}x_1x_n\right) + (\text{terms in } x_2, \dots, x_n)$$

$$= a_{11} \cdot \left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \cdots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + (\text{terms in } x_2, \dots, x_n)$$

Change variables
$$x_1' = x_1 + \frac{a_{12}}{a_{11}}x_2 + \cdots + \frac{a_{1n}}{a_{11}}x_n$$
$$x_2' = x_2$$
$$\vdots$$
$$x_n' = x_n$$

$$Q(x') = a_{11}x_1'^2 + Q_1(x_2', \dots, x_n')$$

Continue the process

Continuing this way, we can rewrite Q

as $a_1 z_1^2 + \cdots + a_n z_n^2$.

$a_1, \ldots, a_n \in \mathbb{R} \longleftrightarrow A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$.

<u>Special Case</u>: Suppose all $a_{ii} = 0$.

Some term has to be nonzero; say $a_{12} \neq 0$

change variables $x_1 = x_1' - x_2'$, $x_2 = x_1' + x_2'$

so $2a_{12}(x_1'^2 - x_2'^2)$ + other terms

<u>Sesquilinear Forms</u>

V a complex space.

$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$

(a) $\langle \alpha \vec{x} + \beta \vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$

(b) $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

<u>Main Example</u>:

$\langle \vec{x}, \vec{y} \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$

$\langle \vec{x}, \vec{x} \rangle \geq 0$ & $= 0 \iff \vec{x} = 0$.

$U(n) = \{ g \in GL(n, \mathbb{C}) \mid \langle g\vec{x}, g\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

$\iff \langle g\vec{x}, g\vec{x} \rangle = \langle \vec{x}, \vec{x} \rangle$

$\iff A^{-1} = {}^T\bar{A}$ ${}^T\bar{A} \cdot A = Id$

$SU(n) = \{ g \in U(n) \mid \det g = 1 \}$.

In general, if $g \in U(n)$,

$${}^t\bar{g} \cdot g = I \Rightarrow \overline{\det g} \cdot \det g = 1$$

$\Leftrightarrow |\det g| = 1$. all numbers of the form $e^{i\theta}$, $\theta \in \mathbb{R}$

Structure of $SU(2)$.

$$\{e_1, e_2\} \longmapsto \{g \cdot e_1, g \cdot e_2\}$$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow g \cdot e_1 = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \quad g \cdot e_2 = \begin{pmatrix} \beta \\ \delta \end{pmatrix}$$

unit vectors

$$\begin{cases} |\alpha|^2 + |\gamma|^2 = 1, & |\beta|^2 + |\delta|^2 = 1 \\ \alpha \bar{\beta} + \gamma \bar{\delta} = 0 \end{cases}$$

$$\alpha\delta - \beta\gamma = 1$$

$$\alpha = r_1 e^{i\theta_1} \qquad \gamma = r_2 e^{i\theta_2}$$

with $r_1^2 + r_2^2 = 1$ so

$$\alpha = \cos\theta \, e^{i\theta_1} \qquad \gamma = \sin\theta \, e^{i\theta_2}$$

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \, e^{i\theta_1} \\ \sin\theta \, e^{i\theta_2} \end{pmatrix}$$

$$= g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2/1/2

$$\underbrace{\begin{pmatrix} \cos\theta + \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{-i\theta_2} \end{pmatrix} g}_{g'} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$g'\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$ unit vector $\perp \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$0 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix} \right\rangle = \beta$$

So $g' \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tau} \end{pmatrix}$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tau} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}$$

If $\det g = 1$,

$$g = \begin{pmatrix} e^{i\tau_1} & 0 \\ 0 & e^{-i\tau_1} \end{pmatrix}\begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} e^{i\tau_2} & 0 \\ 0 & e^{-i\tau_2} \end{pmatrix}$$

3 parameters. <u>NOTE</u>: $SU(2)$ is a <u>real</u> group

<u>Group Homomorphisms</u>

$\pi : G \longrightarrow H$ $\qquad \pi(g_1 g_2) = \pi(g_1)\pi(g_2)$

$\pi(e) = e$.

$G = SU(2)$ vs $H = SO(3)$

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \right\} \quad a,b,c \in \mathbb{R}$$

$$\langle Xv, w \rangle + \langle v, Xw \rangle = 0$$

$${}^t\overline{X} = -X$$

If $g^{-1} = {}^t\overline{g}$, ${}^t\overline{X} = -X$ then

$${}^t\overline{(gXg^{-1})} = -gXg^{-1}.$$

So we get a map

$$G = SU(2) \longrightarrow GL(3)$$

In fact this presentation of $\mathbb{R}^3$ has an inner product which is naturally preserved by the action we defined:

$$\langle X, Y \rangle := tr({}^t\overline{X} Y)$$

$$\begin{pmatrix} ia_1 & b_1+ic_1 \\ -b_1+ic_1 & -ia_1 \end{pmatrix} \cdot \begin{pmatrix} -ia_2 & -b_2-ic_2 \\ b_2-ic_2 & +ia_2 \end{pmatrix} =$$

$$= \begin{pmatrix} +a_1a_2 + b_1b_2 + c_1c_2, & * \\ * & a_1a_2 + b_1b_2 + c_1c_2 \end{pmatrix} =$$

$$= 2(a_1a_2 + b_1b_2 + c_1c_2$$

$$tr(g\bar{X}^T g^{-1} \cdot g Y g^{-1}) = tr(g \bar{X}^T Y g^{-1}) = tr(g^{-1} g \bar{X}^T Y)$$
$$= tr(\bar{X}^T Y).$$

Properties of trace:

$tr(AB) = tr(BA)$

$tr(A+B) = tr A + tr B$

$tr(cA) = c\, tr A.$

So the map

$$SU(2) \longrightarrow GL(3)$$

actually has image in $O(3)$.

In fact the image is in $SO(3)$, determinant is 1.

Example of a homomorphism

G a group

$$Aut(G) := \{ \varphi : G \to G \mid \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2),\ \varphi(e) = e \}$$

This forms a group.

$Adj : G \longrightarrow Aut(G)$ $\quad Adj(g)(h) := g h g^{-1}$

Semidirect Product

G, H groups, $\Psi : G \to Aut(H)$ a group homomorphism.

$G \ltimes H := G \times H$ but multiplication is given by

$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, \Psi(g_2) h_1 \cdot h_2)$

Examples:

(1) $G = (O(3), \cdot)$ $V \simeq (\mathbb{R}^3, +)$

$\Psi : G \to Aut(V)$ $g \mapsto \Psi(g) v = g \cdot v$

$G \ltimes V$ is the ~~a~~ group of rigid motions.

$SO(3) \ltimes V$ is the group of "orientation preserving" rigid motions.

(2) Upper triangular group.

$G : \left\{ \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} ; a_1 \cdot a_2 \neq 0 \right\}$.

$G = A \ltimes \mathbb{R}$ $A := \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1 \cdot a_2 \neq 0 \right\}$

$\mathbb{R} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ with multiplication

$\Psi : A \to Aut(\mathbb{R})$ $\Psi(a) v = a v a^{-1}$

Inner Product on the vector space of $n \times n$ matrices, real or complex.

$V = M_{m \times n} = \{ n \times m$ matrices with coefficients in $F = \mathbb{R}$ or $\mathbb{C} \}$

$X = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

m rows n columns

${}^T X = (a_{ji})$ $n \times m$ matrix.

$\overline{X} = (\overline{a_{ij}})$ $m \times n$ matrix

$\langle\, ,\rangle : V \times V \longrightarrow \mathbb{C}\ (\mathbb{R})$.

$\langle X, Y \rangle = tr(X \cdot {}^T\overline{Y})$

(1) linear in $\langle \cdot\, , \text{fixed} \rangle$

skew linear in $\langle \text{fixed}, \cdot \rangle$

(2) $\langle X, Y \rangle = \overline{\langle Y, X \rangle}$

Follows from the fact that

$\langle X, Y \rangle = \sum_{i,j} a_{ij} \overline{b_{ij}}$

$GL(m,\mathbb{R}) \times GL(n,\mathbb{R})$ <u>acts</u> on $M_{m\times n}$

$(g_1, g_2)\cdot X = g_1 \cdot X \cdot g_2^{-1}$

<u>Proposition:</u> If $g_1^{-1} = {}^T\overline{g_1}$, $g_2^{-1} = {}^T\overline{g_2}$, then

$\langle gX, gY\rangle = \langle X, Y\rangle$

<u>Proof:</u> $(g_1 X g_2^{-1})\cdot {}^T\overline{(g_1 Y g_2^{-1})} =$

$= g_1 X g_2^{-1} \cdot {}^T\overline{(g_2^{-1})} \cdot {}^T\overline{Y} \cdot {}^T\overline{g_1} = g_1 X \cdot {}^T\overline{Y} g_1^{-1}$

Use properties of trace.

<u>REMARK:</u> (1) $(g_1, g_2) \longmapsto \varphi_{(g_1,g_2)} \in Aut(M_{m\times n})$

is a group homomorphism

(2) $GL(m,F)\times GL(n,F) \longrightarrow GL(mn, F)$.

The case $\Phi: SU(2) \longrightarrow SO(3)$

is a bit different; same flavor though.

(3) the proposition says that $g \mapsto \varphi_g$ takes

$O(m)\times O(n) \longrightarrow O(mn)$.

$V = \mathbb{C}^2$

Define $\langle (z_1, z_2), (w_1, w_2) \rangle = z_1 w_1 + z_2 w_2$

What is $SO(\langle\, , \,\rangle)$?

The basis e_1, e_2 satisfies

$(e_1, e_1) = (e_2, e_2) = 1, \quad (e_1, e_2) = 0.$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1$$

$$\alpha^2 + \gamma^2 = 1 \qquad \alpha\beta + \gamma\delta = 0$$

$$\beta^2 + \delta^2 = 1$$

$$\begin{pmatrix} \alpha & -t\gamma \\ \gamma & t\alpha \end{pmatrix} \text{ for some } t; \quad t(\alpha^2 + \gamma^2) = 1 \Rightarrow t = 1$$

$$\begin{pmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{pmatrix} \text{ with } \alpha^2 + \gamma^2 = 1$$

$$\alpha = r_1 e^{i\theta_1} \qquad \gamma = r_2 e^{i\theta_2}$$

$$\alpha = \cos z \qquad \beta = \sin z$$

with z complex.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$$

Image of $SU(2)$ in $O(3)$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} ia' & b'+ic' \\ -b'+ic' & -ia' \end{pmatrix}$$

$$(a, b, c) \longmapsto (a, \cos 2\theta\, b + \sin 2\theta\, c, \cos 2\theta\, b - \sin 2\theta\, c)$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

sends

$$(a, b, c) \longmapsto (\cos 2\theta\, a, b, a \sin 2\theta - c\cos 2\theta)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{pmatrix}, \quad \begin{pmatrix} \cos 2\theta & 0 & \sin 2\theta \\ 0 & 1 & 0 \\ -\sin 2\theta & 0 & \cos 2\theta \end{pmatrix}$$

PROPOSITION: Any $g \in SO(3)$ can be written as

$$\begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

(Exercise)

A group homomorphism

$\pi : G \longrightarrow GL(V)$ V vector space

is called a representation.

Example 1:

$G = SU(n, \mathbb{F}) \longrightarrow GL(M_{n\times n})$

$\pi(g) \cdot X := g X g^{-1}$

Example 2

$\pi : G = GL(n, \mathbb{F}) \longrightarrow GL(M_{n\times n}^{\text{symmetric}})$

$\pi(g) \cdot X = g X g^{T}$

Example 3

$\pi : G = GL(n, \mathbb{F}) \longrightarrow M_{n\times n}^{\text{antisymmetric}}$

In example 1, let $\mathbb{F} = \mathbb{C}$.

$M_{n\times n} \supset M_{n\times n}(\text{skew})$

$SU(n)$ preserves this subspace.

Image is in $O(M^{as}_{n\times n})$

1.) Map is onto

2.) kernel is $\mathbb{Z}_2$

Kernel: $gXg^{-1} = X$ for all X skew

$\Rightarrow gXg^{-1} = X$ for all 2×2 matrices

$\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \Rightarrow$ $b = c = 0$

$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \Rightarrow$ $a = d$

$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\pi : (\mathbb{R}, +) \longrightarrow S^1$ group homomorphism

$\log : U \subseteq S^1 \longrightarrow \mathbb{R}$

$e^{i\theta} \longmapsto \theta$

For

$\varphi : V_1 \subseteq \mathbb{R} \longrightarrow V_2 \subseteq \mathbb{R}$

$\varphi(t_1 + t_2) = \varphi(t_1) + \varphi(t_2)$ for small t_1, t_2

$\varphi : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$

$$\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2)$$

if $x_1, x_2, x_1 + x_2 \in (-\epsilon, \epsilon)$

For any $x \in \mathbb{R}$ there is $n \in \mathbb{N}^+ \ni \left|\frac{x}{n}\right| < \epsilon$

Define $\varphi(x) = n\,\varphi\left(\frac{x}{n}\right)$

2/8/3

$\varphi : \mathbb{R} \to \mathbb{R}$ continuous $\ni$ $\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2)$

$\Rightarrow \exists a \ni \varphi(x) = ax.$

$$\begin{tikzcd} \varphi : \mathbb{R} \arrow[r] \arrow[dr, "\pi"'] & \mathbb{R} \arrow[d, "e^{i\cdot}"] \\ & S^1 \end{tikzcd}$$

commutes.

$SO(2, \mathbb{C})$.

$\langle (x_1, x_2), (y_1, y_2) \rangle := x_1 y_1 + x_2 y_2$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$\alpha^2 + \gamma^2 = 1$

$\beta^2 + \delta^2 = 1$

$\boxed{\alpha\beta + \gamma\delta = 0,}$

$\alpha\delta - \beta\gamma = 1$

$\boxed{\alpha = \cos z, \quad \beta = \sin z}$

$\gamma = -\sin z, \quad \delta = \cos z$

$\alpha\beta = -\gamma\delta$

$\begin{pmatrix} \cos z & -t \sin z \\ \sin z & t \cos z \end{pmatrix} \Rightarrow t = 1.$

2/8/4

$$\langle\!\langle (x_1, x_2), (y_1, y_2) \rangle\!\rangle = x_1 y_2 + y_1 x_2$$

$$(x_1, x_2) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1 y_1 + x_2 y_2)$$

$$(x_1, x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_2, x_1) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \alpha \\ \delta & \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 2\gamma\alpha, & \gamma\beta + \alpha\delta \\ \gamma\beta + \alpha\delta, & 2\beta\delta \end{pmatrix}$$

$$\alpha\gamma = 0, \quad \beta\delta = 0.$$

$$\gamma = 0 \Rightarrow \beta = 0. \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \qquad \alpha\delta = 1.$$

$$\alpha = 0 \Rightarrow \delta = 0 \qquad \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \qquad \beta\gamma = 1$$

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are equivalent

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\underline{SO(4,1)}$ $\langle (x_1 x_2 x_3 x_4), (y_1, y_2, y_3, y_4) \rangle$

$= x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$

$\det g = 1$

$J = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ $\qquad g J {}^T g = J$

$S[O(3) \times O(1)] \subseteq SO(4,1)$

Polar Decomposition

$g \in GL(n, \mathbb{R})$

$g = k \cdot p$ uniquely such that $k \in O(n)$

p symmetric positive definite.

$s = {}^T g g$ is symmetric. $\exists\, p$ symmetric $\ni p^2 = {}^T g g$.

$\langle s x, y \rangle = \langle x, s y \rangle$

1) s has real eigenvalues only

$s \cdot v = \lambda v.$ $\qquad \langle s v, v \rangle = \lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle$

2) s is diagonalizable

$s v = \lambda v$ $\qquad W = (\mathbb{R} v)^{\perp}$ $\qquad sW \subseteq W$

3.) There is a basis of o.n. eigenvectors.

So $\exists\, k$ in $O(n) \ni k s k^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\lambda_i > 0$.

Let $p = k^{-1} \cdot \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix} k$

Then
$$p^2 = g^T g$$

$k = g p^{-1}$ is in $O(n)$

$(g p^{-1})^T \cdot (g p^{-1}) = p^{-1} g^T g \cdot p^{-1} = p^{-1} \cdot p^2 \cdot p^{-1} = I$ ✓

4.)
If $k_1 \cdot p_1 = k_2 \cdot p_2$ then $k_1 = k_2$, $p_1 = p_2$

$s = g^T g = p_1^2 = p_2^2 \Longrightarrow p_1 = p_2$

If $s = p^2$, then s and p commute.

If $p \cdot e = \lambda e$ then $se = \lambda^2 e$, i.e.

any eigenvector of p is an eigenvector of s.

Need the fact that p has positive eigenvalues only.

$V_\mu = \{ v \in V : sv = \mu v \}$ $\quad \mu \geq 0$

$v \in V_\mu \Longrightarrow pv \in V_\mu$; $s(pv) = p(sv) = \mu pv$.

2/8/7

$SL(n,\mathbb{R}) \cong SO(n) \times \mathcal{H}$

Suppose $G = SO(4,1)$.

If $g \in G$, then ${}^Tg \in G$.

$${}^Tg\, J g = J \implies g J\, {}^Tg = J$$

$$J X + {}^T X J = 0$$

$$\langle J X v, w \rangle + \langle J v, X w \rangle = 0$$

$g \in SO(1,1)$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{J_{11}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} = \begin{pmatrix} -c & a \\ d & -b \end{pmatrix} \implies g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad g = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

$${}^Tg\, J g = J \implies g^{-1} J\, {}^Tg^{-1} = J$$

because $J^{-1} = J$.

$$(gv, gw) = (v, w) \implies (g^{-1}v, g^{-1}w) = (v, w)$$

$$\langle J g v, g w \rangle = \langle J v, w \rangle \qquad \langle J g^{-1} v, g^{-1} w \rangle = \langle J v, w \rangle$$

<u>Exercise</u>: $\Phi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +)$

group homomorphism. Then there is $a \in \mathbb{R}$

such that $\Phi(x) = ax$.

<u>Proof</u>: $\Phi(1) = a$, $\Phi(n) = \Phi(\underbrace{1 + \cdots + 1}_{n}) = n\Phi(1) = na$

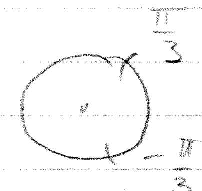
$$n \cdot \Phi\left(\frac{1}{n}\right) = \Phi\left(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n}\right) = \Phi(1) = a$$

$\Phi\left(\frac{1}{n}\right) = a \cdot \frac{1}{n}$ $\quad$ $\Phi\left(\frac{m}{n}\right) = \frac{m}{n}\Phi(1) = \frac{m}{n}a$.

By continuity, $\Phi(x) = ax$.

$$\Phi : (\mathbb{R}, +) \to (S^1, \cdot).$$

$$S^1 \setminus \{-1\} \overset{\sim}{\longleftrightarrow} (-\pi, \pi).$$

Let $\epsilon > 0$ be such that $\Phi(-\epsilon, \epsilon) \subset$ 

Define $\varphi : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$

by $\quad \Phi(x) = e^{i\varphi(x)}$

If $-\epsilon < x_1, x_2, x_1 + x_2 < \epsilon$ then $\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2)$

$\varphi(-x) = \varphi(x) \quad -\epsilon < x < \epsilon$

If x is arbitrary, define $\Psi(x)$ as follows

$\Psi(x) := n\,\varphi\left(\frac{x}{n}\right)$ for any $n \in \mathbb{N} \ni -\epsilon < \frac{x}{n} < \epsilon$

Then Ψ is a homomorphism $(\mathbb{R}, +) \to (\mathbb{R}, +)$

$\Phi(x) = e^{i\Psi(x)}$.

2/10/2

$g \in SO(3)$. Then g has an eigenvalue λ & v

~~Let we~~ $g \cdot v = \lambda v$, $\langle v, v \rangle = 1$.

$1 = \langle v, v \rangle = \langle gv, gv \rangle = \lambda \cdot \bar{\lambda} \langle v, v \rangle$ so $|\lambda|^2 = 1$.

g also has eigenvalue $\bar{\lambda}$.

$g \cdot \bar{v} = \overline{gv} = \overline{\lambda v} = \bar{\lambda} v$.

If λ is real, $\lambda = \pm 1$.

If not, $\lambda \neq \bar{\lambda}$ so v and $\bar{v}$ are lin. independent

$v + \bar{v}$ and $i(v - \bar{v})$ are indep. & in $\mathbb{R}^3$

$\lambda = \cos\theta + i \sin\theta$, $\bar{\lambda} = \cos\theta - i\sin\theta$

If v_1, v_2 are eigenvectors with eigenvalues $\lambda_1 \neq \lambda_2$, then $\langle v_1, v_2 \rangle = 0$. If not,

$$\langle v_1, v_2 \rangle = \langle g v_1, g v_2 \rangle = \lambda_1 \bar{\lambda}_2 \langle v_1, v_2 \rangle = \frac{\lambda_1}{\lambda_2} \langle v_1, v_2 \rangle$$

So $v, \bar{v}$ are orthogonal. So are

$$\frac{v + \bar{v}}{\sqrt{2}}, \quad \frac{(v - \bar{v})}{\sqrt{2} i}.$$

$$g \cdot \left(\frac{v + \bar{v}}{\sqrt{2}} \right) = \frac{\lambda v + \bar{\lambda} \bar{v}}{\sqrt{2}} = \cos\theta \, \frac{v + \bar{v}}{\sqrt{2}} - \sin\theta \, \frac{v - \bar{v}}{i\sqrt{2}}$$

$$g \left(\frac{v - \bar{v}}{\sqrt{2} i} \right) = \frac{\lambda v - \bar{\lambda} \bar{v}}{\sqrt{2} i} = + \sin\theta \, \frac{v + \bar{v}}{\sqrt{2}} + \cos\theta \, \frac{v - \bar{v}}{i\sqrt{2}}$$

Look at the vector $e \perp$ to $\frac{v+\bar{v}}{\sqrt{2}}$, $\frac{v-\bar{v}}{\sqrt{2}i}$.

ge is $\perp$ $\frac{v+\bar{v}}{\sqrt{2}}$, $\frac{v-\bar{v}}{\sqrt{2}i}$ so $ge = me$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \ \det = 1 \Rightarrow m = 1.$$

If g has only real eigenvalues, they are ± 1.

$-1, -1, -1$ gives determinant 1.

<u>Second Proof:</u> $\det g = 1$

${}^T g = g^{-1}$. Eigenvalues $p_g(\lambda) = \det(\lambda I - g)$

$$= \lambda^3 + a_2\lambda^2 + a_1\lambda - 1$$

$$\det(\lambda I - g^{-1}) = \det((\lambda g - I)g^{-1}) = \det(\lambda g - I)$$

$$= \det((-\lambda)(\tfrac{1}{\lambda} I - g)) = (-\lambda)^3 p_g(\tfrac{1}{\lambda}).$$

$$= -\lambda^3\left(\frac{1}{\lambda^3} + a_2\frac{1}{\lambda^2} + a_1\frac{1}{\lambda} - 1\right) = \lambda^3 - a_1\lambda^2 - a_2\lambda - 1$$

$$\det(\lambda I - g^{-1}) = \det(\lambda I - {}^T g) = \det((\lambda I - g)^T) = \det(\lambda I - g)$$

$$\lambda^3 + a_1\lambda^2 + a_2\lambda - 1 = \lambda^3 - a_2\lambda^2 - a_1\lambda - 1$$

So $a_1 = -a_2$ $\quad (\lambda^3 + a\lambda^2 - a\lambda - 1)$

$$= (\lambda-1)(\lambda^2+\lambda+1) + a\lambda(\lambda-1) = (\lambda-1)(\lambda^2 + (a+1)\lambda + 1)$$

$$\frac{-(a+1) \pm \sqrt{(a+1)^2 - 4}}{2}$$

SU(2) has a 3-dimensional rep'n.

$$\pi_3 : SU(2) \longrightarrow GL(3,\mathbb{C})$$

$$\mathbb{C}^3 \simeq \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

$$\pi_3(g)\, X := g X g^{-1}$$

also $\pi : SU(2) \longrightarrow GL(2,\mathbb{C})$

$$\pi_2(g) v = g \cdot v$$

A representation π is called irreducible, if any

$\pi : G \longrightarrow GL(V)$

V real or cx

$\boxed{V \text{ cx}}$

$$W \leq V \text{ satisfying } \pi(g) W \subset W \Rightarrow W = (0) \text{ or } W = V$$

unitarizable if $\exists \; \langle , \rangle$ s. that

$$\langle \pi(g) v, \pi(g) w \rangle = \langle v, w \rangle$$

Here are more such representations:

$$V_n = \operatorname{span}\{ x^a y^b : a+b=n \} = \{ P : \mathbb{C}^2 \to \mathbb{C} \text{ homogeneous of degree } n \}$$

$g \in SL(2,\mathbb{C})$

$$(\pi_n(g) P)(v) = P(g^{-1} v)$$

$$\pi_n(g_1 g_2) P(v) = P((g_1 g_2)^{-1} v) = P(g_2^{-1} g_1^{-1} v) =$$

$$= (\pi_n(g_2) P)(g_1^{-1} v) = \pi_n(g_1)(\pi_n(g_2) P)(v)$$

$$(x^k \cdot y^l)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha^k \cdot \beta^l$$

$$\pi_n\begin{pmatrix}a & b\\ c & d\end{pmatrix}(x^k y^l)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = x^k y^l\left(\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)$$

$$= x^k y^l\begin{pmatrix}d\alpha - b\beta\\ -c\alpha + a\beta\end{pmatrix} = (d\alpha - b\beta)^k(-c\alpha + a\beta)^l$$

$$\pi_1(g)(x)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = d\alpha - b\beta \qquad dx - by$$

$$\pi_1(g)(y)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -c\alpha + a\beta \qquad -cx + ay$$

$$\pi_n(g)\cdot(x^k y^l) = (dx - by)^k(-cx + ay)^l$$

$$\pi_n\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}(x^k y^l) = (-x)^k \cdot (-y)^l = (-1)^{k+l} x^k y^l = (-1)^n x^k y^l$$

n even $\pi_n\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = I_v$

Then V_n drops to $SO(3)$

Exponential Map

$X \in M_{n\times n}(\mathbb{C})$.

$$|X| = \left(\sum_{i,j} |x_{ij}|^2\right)^{\frac{1}{2}}$$

Comes from $\langle X, Y\rangle := \mathrm{tr}(X \cdot {}^T\bar{Y})$

Properties:

$$|X+Y| \leq |X| + |Y|$$

$$|X \cdot Y| \leq |X| \cdot |Y|$$

Proof:

$$\langle X+Y, X+Y\rangle = \langle X, X\rangle + \langle X, Y\rangle + \langle Y, X\rangle + \langle Y, Y\rangle$$

But $|\langle X, Y\rangle| \leq |X| \cdot |Y|$ (Cauchy-Schwartz)

so

$$\langle X+Y, X+Y\rangle \leq (|X| + |Y|)^2.$$

" Do exercise 1 section 2 "

Form the series

$$\exp X := \sum_k \frac{X^k}{k!} .$$

This series converges for any X. It is "absolutely convergent, because

$$|X^k| \leq |X|^k.$$

Consider $\underset{X}{f}(t) = \exp(tX)$. Then

$$\frac{df}{dt} = X \exp tX = \exp tX \cdot X.$$

We can use this to solve differential equations. Let A be an $n \times n$ matrix, $\vec{x} = (x_i)$ a column vector. A system of

diff. equations:

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \vec{x}(0) = x_0$$

$$\vec{x} = \exp(tA) \cdot \vec{x}_0.$$

Examples:

(1) X upper triangular. Series stops after finitely many terms because $X^n = 0$

(2) X diagonal or diagonalizable

(3) $X = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \iff \frac{dx_1}{dt} = -x_2 \quad \frac{dx_2}{dt} = x_1$

$X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad X^3 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \quad X^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

2/15/3

<u>Properties:</u> $\exp(0) = I$

$$(\exp X)^* = \exp X^*$$

$$\exp(-X) = (\exp X)^{-1}$$

$$\exp X \cdot \exp Y = \exp(X+Y) \quad \text{if } XY = YX$$

$$g \exp X g^{-1} = \exp(gXg^{-1})$$

$$|e^X| \le e^{|X|}.$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} & \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} & \end{pmatrix}.$$

<u>Definition:</u> $\exp tX$ is called a 1-parameter group.

It is a homomorphism (continuous)

$$(\mathbb{R}, +) \longrightarrow GL(V) \qquad V = \mathbb{C}^n.$$

<u>Logarithm:</u> $\log z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(z-1)^k}{k}$

Converges for $|z-1| < 1$. $\quad e^{\log z} = z$

If $|u| < \log 2$, $\quad |e^u - 1| < 1$ and

$\log e^u = u.$

If X is a matrix satisfying $|X-I|<1$,
we can form $\log X$.

$$\exp(\log X) = X$$

Note that X is invertible.

If $|X| < \log 2$, $|\exp X - I| < 1$ and

$$\log(\exp X) = X.$$

Properties of exp:

$\exp(0) = I$

$\exp(X)^* = \exp(X^*)$

$\exp X \cdot \exp Y = \exp(X+Y)$ if $XY = YX$

In particular

$\exp aX \cdot \exp bX = \exp(a+b)X$

$g \exp X g^{-1} = \exp gXg^{-1}$

$\det(\exp X) = \exp \operatorname{tr} X$

$|\exp X| \leq e^{|X|}$.

$\exp \log u = u \qquad \log(\exp X) = X$

for appropriate X and u

$\exp : M_{n\times n} \longrightarrow GL(n)$

is continuous.

$$\exp\left(\frac{X}{n}\right) \cdot \exp\frac{Y}{n}$$

$$\left(I + \frac{X}{n} + \frac{1}{n^2} a_X\right) \cdot \left(I + \frac{Y}{n} + \frac{1}{n^2} b_Y\right)$$

where

$|a_X| \le M_X$, $|b_X| \le M_Y$.

bounds depending on X, Y but <u>not</u> n.

$$I + \frac{X+Y}{n} + \frac{1}{n^2} a(X,Y)$$

$|a(X,Y)| \le M_{X,Y}$ indep. of X, Y.

If n is large enough, we can take log.

$$\exp\left(\frac{X}{n}\right) \exp\left(\frac{Y}{n}\right) = \exp\left(\frac{X+Y}{n} + \frac{1}{n^2} C_{X,Y}\right)$$

Now consider

$$\left(\exp\frac{X}{n} \cdot \exp\frac{Y}{n}\right)^n = \exp\left(X + Y + \frac{1}{n} C_{X,Y}\right)$$

$$\therefore \lim_{n\to\infty} \left(\exp\frac{X}{n} \cdot \exp\frac{Y}{n}\right)^n = \exp(X+Y).$$

Similar argument yields

$$\left(\exp\frac{X}{n} \exp\frac{Y}{n} \exp-\frac{X}{n} \exp-\frac{Y}{n}\right)^{n^2} \longrightarrow \exp(XY - YX).$$

Let $G \subseteq GL(V)$ be a matrix group, i.e. a closed subgroup. Define

$$\mathcal{L}(G) := \{X \in gl(V) : \exp tX \in G \ \forall t \in \mathbb{R}\}$$

<u>THEOREM</u>: $\mathcal{L}(G)$ is a Lie algebra.

<u>Definition</u>: Call $gl(V)$ the space of linear transformations of the vector space V.

$$[\cdot,\cdot] : gl(V) \times gl(V) \longrightarrow gl(V)$$

$$[X,Y] := XY - YX$$

It satisfies

(a) $[aX+bY, Z] = a[X,Z] + b[Y,Z]$

(b) $[X,Y] = -[Y,X]$

(c) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

<u>Definition</u>: A Lie algebra is a vector space g with a $[\cdot,\cdot] : g \times g \longrightarrow g$ satisfying (a) - (c).

<u>Examples</u>: $SU(n)$, $SL(n)$, $O(n)$, $Sp(n)$.

2/17/4

Notes on the homework:

Quadratic Forms:

V a vector space, coordinates $x_1, \dots, x_n$

$$Q(x) = \sum a_{ij} x_i x_j \qquad a_{ij} = a_{ji}$$

is called a quadratic form.

$\Longleftrightarrow$

$$(v, w)_Q := Q(v+w) - Q(v) - Q(w)$$

is bilinear symmetric

If $Q(x) = \sum a_i^2 x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j$

let $A = (a_{ij})$ be the corresponding matrix.

Write $(v, w) = ({}^T v A w)$ $\quad$ 1×1 matrix scalar

v, w column matrices

$Q(x) = {}^T x A x$.

BASIC PROBLEM: Find a basis where a given $Q(x)$ is as simple as possible (often done over a ring rather than a field)

2/17/5

$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

MUST HAVE $\det g = 1$.

$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} e^t\cos\theta_1 & -e^t\sin\theta_1 \\ e^{-t}\sin\theta_1 & e^{-t}\cos\theta_1 \end{pmatrix} =$$

$$\begin{pmatrix} e^t\cos\theta_1\cos\theta_2 - e^{-t}\sin\theta_1\sin\theta_2, & -e^t\sin\theta_1\cos\theta_2 - e^{-t}\sin\theta_2\cos\theta_1 \\ e^t\cos\theta_1\sin\theta_2 + e^{-t}\sin\theta_1\cos\theta_2, & -e^t\sin\theta_1\sin\theta_2 + e^{-t}\cos\theta_1\cos\theta_2 \end{pmatrix}$$

So

$$a+d = (e^t+e^{-t})\cos(\theta_1+\theta_2)$$
$$-b-c = -(e^t+e^{-t})\sin(\theta_1+\theta_2)$$

$$a-d = (e^t-e^{-t})\cos(\theta_2-\theta_1)$$
$$-b+c = (e^t-e^{-t})\sin(\theta_2+\theta_1)$$

Check that you can solve.

CONFORMAL TRANSFORMATIONS OF THE UPPER HALF PLANE

$$\mathcal{H} := \{z : \operatorname{Im} z > 0\} \cup \{\infty\}$$

$SL(2,\mathbb{R})$ acts on $\mathcal{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot z = \frac{az+b}{cz+d}$$

satisfies $\pi(g_1 g_2) = \pi(g_1)\cdot\pi(g_2)$.

Stabilizer of $i = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\cdot i = \frac{\cos\theta e^{2t} i - \sin\theta}{\sin\theta e^{2t} i + \cos\theta} =$$

$$= \frac{(-\sin\theta + i e^{2t}\cos\theta)\cdot(\cos\theta - i\sin\theta e^{2t})}{\cos^2\theta + e^{4t}\sin^2\theta}$$

$$x = \frac{-\cos\theta\sin\theta + \cos\theta\sin\theta e^{4t}}{\cos^2\theta + e^{4t}\sin^2\theta}$$

$$y = \frac{\cos^2\theta e^{2t} + \sin^2\theta e^{2t}}{\cos^2\theta + e^{4t}\sin^2\theta}$$

$$\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} i = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}(x+i) =$$

$$\frac{e^{t}(x+i)}{e^{-t}} = e^{2t}x + e^{2t}i$$

easy to solve

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & e^{-2t}x \\ 0 & 1 \end{pmatrix}.$$

One parameter subgroups:

$\pi : (\mathbb{R}, +) \longrightarrow GL(m, \mathbb{C})$

a representation which is continuous.

$\pi(0) = Id$

$\pi(t+s) = \pi(t) \cdot \pi(s)$

$\pi(-t) = \pi(t)^{-1}$.

Proposition: There is $\overset{\text{a unique}}{X} \in gl(m, \mathbb{C})$ such that

$$\pi(t) = \exp(tX).$$

Proof: $\frac{d}{dt}\Big|_{t=0} \exp(tX) = X$ shows uniqueness

Existence: Let $\epsilon > 0$ be such that

$\log : \mathcal{U}_\epsilon \longrightarrow \mathcal{V} \subseteq gl(m, \mathbb{C})$ is well defined and

$\exp \circ \log = id$, $\log \circ \exp = id$.

There is $\delta > 0$ so that $\pi(-\delta, \delta) \subseteq \mathcal{U}_\epsilon$.

$\exists X \ni$ $\log \circ \pi(t) = tX$ in this neighborhood.

$\pi(t) = \exp tX$

for $-\delta < t < \delta$. For arbitrary t,