3.10. Proposition. The component of the identity G_0 of a Lie group G is a Lie subgroup with respect to the standard C^r-structure induced on the open submanifold G_0.

Proof. Obvious by the definition of the C^r-structure of G_0

3.11. Definition. Let G, G' be Lie groups. A mapping $F: G \to G'$ is called a Lie homomorphism or simply a homomorphism, if F is a C^r-map and also a group homomorphism, i.e. $F(gh) = F(g)F(h)$ for all $g, h \in G$. If, in addition, F is a C^r-diffeomorphism, it is called a Lie isomorphism or an isomorphism.

If $F: U \to G'$ is a mapping defined on a neighborhood of the identity in G with the property that for some neighborhood V of the identity such that $VV \subset U$ we have $F(gh) = F(g)F(h)$ for all $g, h \in V$, then F is called a local Lie homomorphism if it is also a C^r-map on the open submanifold U of G. It is called a local Lie isomorphism, if it is a C^r-diffeomorphism of U onto an open neighborhood of the identity in G'.

3.12 Lemma. Let G, G' be Lie groups with G connected. Let U be a neighborhood of the identity in G and $F: U \to G'$ a local Lie homomorphism. Then there exists at most one Lie homomorphism $\hat{F}: G \to G'$ which coincides with F on some neighborhood V of e, $V \subset U$.

~~Proof. Let V be a neighborhood of the identity in G such that $VV \subset U$. Let $g \in G$; since V generates G, we can write $g = g_1 \cdots g_n$, with $g_i \in V$.~~

Proof. Since G is connected, it is generated by V. Let $g \in G$ and $g = g_1 \dots g_n$, with $g_i \in V$. Then by definition

$$\hat{F}(g) = \hat{F}(g_1)\dots\hat{F}(g_n) = F(g_1)\dots F(g_n)$$

which proves that the value of $\hat{F}(g)$, if $\hat{F}$ exists, is completely determined by the values of F on U.

3.13. Lemma. Let G, G' be Lie groups and $F: G \to G'$ a group homomorphism (not necessarily continuous). If F is a C^r-map on some neighborhood U of the identity e of G, then F is a Lie homomorphism.

Proof. Let $g_0 \in G$ and $g \in L_{g_0}(U)$. Then

$$F(g) = F(g_0 g_0^{-1} g) = F(g_0)\, F(g_0^{-1} g)$$

and the map $L_{g_0}(U) \xrightarrow{L_{g_0^{-1}}} U \xrightarrow{F} F(U) \xrightarrow{L_{F(g_0)}} F(g_0)F(U)$ is obviously a C^r-map. Thus F is a C^r-map on $L_{g_0}(U)$, which is an open set containing g_0.

3.14. Definition. Let X be an arcwise connected topological space, i.e. for any pair $x_1, x_2 \in X$ there is a continuous curve $\gamma: [0,1] \longrightarrow X$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$. Then X is called simply connected if for any pair $x_1, x_2 \in X$ and any pair of curves γ, δ with endpoints x_1 and x_2 there exists a homotopy between γ and δ keeping the endpoints fixed, i.e. there is a continuous map $H: [0,1]\times[0,1] \longrightarrow X$ such that $H(t,0) = \gamma(t)$, $H(t,1) = \delta(t)$ for all $t \in [0,1]$ and $H(0,s) = x_1$, $H(1,s) = x_2$ for all $s \in [0,1]$. Recall that homotopy is an equivalence relation and that the <u>homotopy class</u> of a curve γ is the equivalence class $[\gamma]$ of γ with respect to homotopy in X with fixed endpoints.

3.15. <u>Theorem</u>. Let G, G' be Lie groups, U a neighborhood of the identity in G and $F: U \to G'$ a local Lie homomorphism. If G is simply connected, then there exists a unique Lie homomorphism $\hat{F}: G \to G'$ such that F and $\hat{F}$ coincide on some neighborhood of the identity.

<u>Proof</u>. We shall extend F to an algebraic homomorphism $\hat{F}: G \to G'$. It follows then from 3.12 and 3.13 that $\hat{F}$ is a C^r-map and unique.

Let V_0 be a connected neighborhood of the identity e in G such that $V_0 V_0 \subset U$ and $F(gh) = F(g)F(h)$ for $g, h \in V_0$. Let $V \subset V_0$ be a connected neighborhood of e such that $V^{-1}V \subset V_0$. Consider the interval $[0,1]$, a continuous curve $\gamma: [a,b] \to G$, and a partition of $[a,b]$:

$$a_0 = a < a_1 < \cdots < a_n = b$$

with the property that $t_1, t_2 \in [a_{i-1}, a_i]$ implies $[\gamma(t_1)]^{-1}\gamma(t_2) \in V_0$. The existence of such a partition follows from the fact that $\gamma(t_1)^{-1}\gamma(t_1) = e$ and the continuity of the group operations, the compactness of $[0,1]$. Set $\gamma_i = [\gamma(a_{i-1})]^{-1}\gamma(a_i) \in V_0$ and define

$$K(\gamma) = F(\gamma_1) \cdots F(\gamma_n) \in G'.$$

The point $K(\gamma)$ does not depend on the choice of the partition. Indeed, for $a_0 = a < a_1 < \cdots < a_{i-1} < c < a_i < \cdots < a_n = b$ we get

$$\gamma_i = [\gamma(a_{i-1})]^{-1}\gamma(a_i) = \gamma(a_{i-1})^{-1}\gamma(c)[\gamma(c)]^{-1}\gamma(a_i) = \gamma_i' \gamma_i''$$

and hence $F(\gamma_i) = F(\gamma_i')F(\gamma_i'')$. Since any two partitions have a common refinement, the statement is proved.

$K(\gamma)$ has the following properties, which are immediate:

(a) for any $g_0 \in G$, $K(g_0\gamma) = K(\gamma)$ (since $(g_0h_1)^{-1}(g_0h_2) = h_1^{-1}h_2$)

(b) for any two curves γ_1, γ_2 in G (such that $\gamma_1(b) = \gamma_2(a)$), $K(\gamma_1 * \gamma_2) = K(\gamma_1)K(\gamma_2)$, where $\gamma_1 * \gamma_2$ is the curve obtained by traversing first γ_1, then γ_2, with matching point.

(c) if $\operatorname{Im}\gamma \subset V$ then $K(\gamma) = F(\gamma(a))^{-1}F(\gamma(b))$

(d) if $\operatorname{Im}\gamma = g_0 \in G$, then $K(\gamma) =$ identity of G!

$K(\gamma)$ depends only on the homotopy class of γ (with fixed endpoints). Let $H: [a,b]\times[0,1] \to G$ be a homotopy between γ_0 and γ_1. The same argument as before implies the existence of two partitions $a_0 = a < a_1 < \dots < a_n = b$ and $0 = \sigma_0 < \sigma_1 < \dots < \sigma_n = 1$ such that $t_1, t_2 \in [a_{i-1}, a_i]$ and $s_1, s_2 \in [\sigma_{i-1}, \sigma_i]$ imply $[H(t_1,s_1)]^{-1}H(t_2,s_2) \in V_0$. We prove now that for any $k = 1, 2, \dots, n$:

$$F\left(H_{\sigma_0}(a_0)^{-1}H_{\sigma_0}(a_1)\right)\dots F\left(H_{\sigma_0}(a_{k-1})^{-1}H_{\sigma_0}(a_k)\right) = \tag{3.15.1}$$

$$= F\left(H_{\sigma_1}(a_0)^{-1}H_{\sigma_1}(a_1)\right)\dots F\left(H_{\sigma_1}(a_{k-1})^{-1}H_{\sigma_1}(a_k)\right)F\left[H_{\sigma_1}(a_k)^{-1}H_{\sigma_0}(a_k)\right]$$

Indeed, for $k=1$ obvious since $H_{\sigma_1}(a_0) = H_{\sigma_0}(a_0)$. For $k+1$ we multiply both sides by

$$F\left(H_{\sigma_0}(a_k)^{-1}H_{\sigma_0}(a_{k+1})\right) = F\left[H_{\sigma_0}(a_k)^{-1}H_{\sigma_1}(a_k)\right]F\left[H_{\sigma_1}(a_k)^{-1}H_{\sigma_1}(a_{k+1})\right]\times$$
$$\times F\left[H_{\sigma_1}(a_{k+1})^{-1}H_{\sigma_0}(a_{k+1})\right]$$

where the equality holds since each product in square brackets lies in V_0. Thus (3.15.1) holds for $k=n$ and since $H_{\sigma_0}(a_n) = H_{\sigma_1}(a_n)$, we have $K(\gamma_0) = K(H_{\sigma_1})$. Repeating this reasoning, we obtain

$K(\gamma_0) = K(\gamma_1)$. Since G is simply connected, all continuous curves in G joining two given points are homotopic. Therefore we have a well-defined mapping $K : G \times G \to G'$ by setting $K(g_1, g_2) = K(\gamma)$, where γ is a curve joining g_1 and g_2.

Define now $\hat{F}(g) = K(e, g)$ for any $g \in G$. Then $g, h \in G$ implies

$$\hat{F}(gh) = K(e, gh) = K(e, g) K(g, gh) = K(e, g) K(e, h) = \\ = \hat{F}(g) \hat{F}(h).$$

Finally, for $g \in V$ there exists a curve joining e and g which is contained in V, since V is connected and therefore arcwise connected. But then $\hat{F}(g) = F(e) \cdot F(g) = F(g)$, which proves that F and $\hat{F}$ coincide on V.

3.16. Corollary. Locally isomorphic simply connected Lie groups are isomorphic.

Proof. Let G, G' be Lie groups, U, V neighborhoods of the identity in G, G' respectively and $F : U \to V$ a local isomorphism. Let $\hat{F} : G \to G'$ and $\hat{F}' : G' \to G$ be the unique extensions of F and of its inverse. Then for some neighborhood of the identity W in G we have $\hat{F}'(\hat{F}(g)) = g$ for $g \in W$. Thus $\hat{F}' \circ \hat{F}$ coincides with the identity mapping on W, hence by uniqueness (Lemma 3.12) $\hat{F}' \circ \hat{F} = id_G$. Similarly we get $\hat{F} \circ \hat{F}' = id_{G'}$.

We have thus shown that a simply connected Lie group is completely determined by its local structure. Our next aim is to show that there is a 1-1 correspondence between the family of classes of locally isomorphic Lie groups" and the family of simply connected Lie groups.

3.17. Definition. Let X be a topological space, $x_0 \in X$. A closed curve in X at x_0 is a continuous mapping $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$. The product of two closed curves (γ, δ) at x_0 is the curve $\gamma * \delta$ defined by

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & , 0 \le t \le \frac{1}{2} \\ \delta(2t-1) & , \frac{1}{2} \le t \le 1 \end{cases}$$

The inverse of γ is the curve γ^- defined by

$$\gamma^-(t) = \gamma(1-t) \, , \quad 0 \le t \le 1 .$$

3.18. Lemma. Let γ be a closed curve in a topological space X at a point x_0 and $[\gamma]$ be the class of all closed curves at x_0 that are homotopic to γ (with fixed endpoints). Then the equations $[\gamma] * [\delta] = [\gamma * \delta]$ and $[\gamma]^- = [\gamma^-]$ define a group structure on the set of all equivalence classes of curves at x_0 with identity $e = [x_0]$, where $[x_0]$ denotes the homotopy class of the constant map.

Proof. Exercise

3.19. Definition. The group defined in 3.18 is denoted by $\pi_1(X, x_0)$ and called the fundamental group of X at x_0.

3.20. Lemma. If X is arcwise connected, then for any $x, y \in X$ the groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic. Moreover, X is simply connected if and only if $\pi_1(X, x) = [x]$ for each $x \in X$.

Proof. Exercise (see verso)

3.21. Lemma. Let X, Y be topological spaces, $f: X \to Y$ a continuous mapping, $x_0 \in X$, $y_0 = f(x_0) \in Y$. Define

$$f_\#: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by $f_\#[\gamma] = [f \circ \gamma]$. Then $f_\#$ is well defined and is a homomorphism. Moreover, if $g: Y \to Z$ is continuous and $z_0 = g(y_0)$, then $(gf)_\# = g_\# f_\# : \pi_1(X, x_0) \to \pi_1(Z, z_0)$. Also $(id_X)_\# = id_{\pi_1(X, x_0)}$.

Proof. Exercise

3.22 Definition. A continuous mapping $p: Y \to X$ is called a covering of X if for each $x \in X$ there exists a simply connected neighborhood U of x such that $p^{-1}(U) \neq \phi$ and the restriction of $p^{-1}(U)$ to every component of $p^{-1}(U)$ is a homeomorphism onto U. The neighborhood U is called distinguished.

3.23. Remark. It is obvious that $p^{-1}(x) \neq \phi$ for every $x \in X$. Moreover, the set $p^{-1}(x)$ is discrete since for every $y \in p^{-1}(x)$ the component of $p^{-1}(U)$ containing y is open and does not contain any other point of $p^{-1}(x)$. It is also clear that card $p^{-1}(x)$ is constant on every component of X, since it is locally constant (and hence continuous into the discrete set of cardinals)

3.24. Example $p: R \to S^1$ with $p(y) = e^{2\pi i y}$

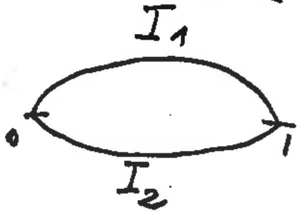
3.25. Proposition. Let $p: Y \to X$ be a covering and γ a curve in X, $y_0 \in p^{-1}(\gamma(0))$. There exists a unique curve $\delta: [0,1] \to Y$ such that $p \circ \delta = \gamma$ and $\delta(0) = y_0$.

Proof. (a) Uniqueness: Suppose there is a $\delta': [0,\tau] \to Y$ $(\tau \leq 1)$ with $p \circ \delta' = \gamma$ and $\delta'(0) = y_0$. Consider the set $A = \{t \in [0,\tau] \mid \delta(t) = \delta'(t)\}$. This set is obviously nonempty and closed. To show that it is open in $[0,\tau]$, take $t_1 \in A$ and consider a distinguished neighborhood U of $p\delta(t_1) = p\delta'(t_1) = x_1$. Let J be an interval containing t_1 such that $p\delta(t), p\delta'(t) \in U$ for $t \in J$. Let $y_1 = \delta(t_1) = \delta'(t_2)$. Since $\operatorname{Im} \delta|_J \cup \operatorname{Im} \delta'|_J$ is connected, it lies in the component of $p^{-1}(U)$ that contains y_1. But p restricted to this component is a homeomorphism onto U, therefore $p \circ \delta|_J = p \circ \delta'|_J$, implies $\delta|_J = \delta|_J'$. Thus A is open in $[0,1]$ and hence $A = [0,\tau]$.

~~(b) Existence. For every $t \in [0,1]$, let U_t be a distinguished neighborhood of $\gamma(t) \in X$. Let B be the set of points $t \in [0,1]$ such that there exists a curve $\delta_t: [0,t] \to Y$ with the properties stated in the theorem. It is nonempty and open; p restricted to the component of $p^{-1}(U_t)$ containing $\delta_t(t)$ is a homeomorphism, hence there is an interval J_t around t such that $\gamma|_{J_t}$ can be lifted.~~

① Proof of : $\pi_1(X,x) = [x] \Rightarrow X$ simply connected.

Let $\gamma_1, \gamma_2 : I \longrightarrow X$ be curves, $\gamma_1(0) = \gamma_2(0) = x$, $\gamma_1(1) = \gamma_2(1)$.
Let S^1 be the space obtained by taking two copies of $I = [0,1]$ and identifying the respective endpoints. Then we have



$$\gamma_1 * \gamma_2^- : S^1 \longrightarrow X$$

We show that $\exists\, \Gamma : D^2 \longrightarrow X$, where D^2 is the 2-cell, such that $\Gamma / S^1 = \gamma_1 * \gamma_2^-$. This will imply the statement, since there exists a homotopy $H : I \times I \longrightarrow D^2$ with $H(t,0) = (\cos \pi t, \sin \pi t)$ and $H(t,1) = (\cos \pi(t+1), \sin \pi(t+1))$. Taking then $h = \Gamma \circ H : I \times I \longrightarrow X$, we see that $h(t,0) = \gamma_1$, $h(t,1) = \gamma_2$.

To prove the existence of Γ, consider a homotopy $F : I \times I \longrightarrow X$ between $\gamma_1 * \gamma_2^-$ and x, whose existence follows from the hypothesis. The homotopy has fixed endpoints, hence $F(0,s) = F(1,s) = x$. Therefore F induces a homotopy $\tilde{F} : S^1 \times I \longrightarrow X$. But $\tilde{F}(S^1 \times \{1\}) = x$, thus we get $\Gamma : C(S^1) \longrightarrow X$, where $C(S^1)$ is the cone over S^1.
But $C(S^1)$ is homeomorphic to D^2.

For every $t \in [0,1]$ let U_t be a distinguished nbhd. of $\gamma(t) \in X$.

(b) Existence: Let B be the union of all the intervals $[0,\tau]$, $\tau \le 1$, with the property that there exists a $\delta_\tau : [0,\tau] \to Y$ satisfying the conditions $p \circ \delta_\tau = \gamma(t)$ for all $t \in [0,\tau]$ and $\delta(0) = y_0$. Obviously B is nonempty: It is open in $[0,1]$ since $\sigma \in B$ implies that there is a component of $p^{-1}(U_\sigma)$ containing $\delta_\sigma(\sigma)$; let q be the restriction of p to this component. Then q is a homeomorphism onto U_σ and there is an interval J_σ around σ such that $\operatorname{Im} \gamma|_{J_\sigma} \subset U_\sigma$. Let $\rho > \sigma$, $\rho \in J_\sigma$ and

$$\delta_\rho(t) = \begin{cases} \delta_\sigma(t), & \text{for } 0 \le t \le \sigma \\ q^{-1}\gamma(t), & \text{for } \sigma \le t \le \rho \end{cases}$$

Then δ_ρ satisfies the above conditions and hence $\rho \in B$.

A similar proof shows that B is closed, hence $B = [0,1]$.

Finally, uniqueness implies that $\delta(t) = \delta_t(t)$ defines a mapping $\delta : [0,1] \to Y$ with the required properties.

3.26. Proposition. Let $p: Y \to X$ be a covering and γ_1, γ_2 be homotopic curves in X (with fixed endpoints). Let δ_1, δ_2 be lifts of γ_1, γ_2 respectively, with $\delta_1(0) = \delta_2(0)$. Then δ_1 and δ_2 are homotopic in Y (with fixed endpoints).

Proof. Let $h : [0,1]^2 \to X$ be the homotopy connecting γ_1 and γ_2. For every $s \in [0,1]$ consider the curve $h_s : [0,1] \to X$ and its unique lifting $H_s : [0,1] \to Y$ such that $H_s(0) = \delta_1(0) = \delta_2(0)$. Thus we get a mapping $H : [0,1]^2 \to Y$, which is continuous for the following reason: let $s_0, t_0 \in [0,1]$ and U be a distinguished neighborhood of the point $h(s_0,t_0)$ in X. Consider the component V of $p^{-1}(U)$ containing the point $H(t_0,s_0)$ and let $W \subset V$ be a neighborhood of this point. Then there are intervals J_{s_0}, J_{t_0} around s_0 and t_0 respectively such that $h(J_{t_0} \times J_{s_0}) \subset p(W)$, which is obviously a neighborhood of $h(s_0,t_0)$. Consider now the points $H(t,s)$ for

Let B denote the set of all pairs $(t,s)\in[0,1]\times[0,1]$ such that H restricted to $[0,t]\times[0,s]$ is continuous. It is nonempty, since $(0,0)\in B$. Indeed, choose a distinguished neighborhood U of $\gamma_1(0)=\gamma_2(0)$ in X and consider the component V of the set $p^{-1}(U)$ that contains the point $\delta_1(0)=\delta_2(0)$. Then for t,s sufficiently close to 0 (i.e. such that $h(t,s)\in U$) we have $H(t,s)=q^{-1}h(t,s)$, where q is the restriction of p to V. This follows from the uniqueness property of path-lifting, since ~~for every~~ fixed s, both sides are continuous liftings of h_s with common initial point $\delta_1(0)=\delta_2(0)$. Thus $H(t,s)$ is continuous in a neighborhood of $(0,0)$.

A similar argument then shows that B is open and closed in $[0,1]\times[0,1]$.

Finally, $H(1,s)=\delta_1(1)$ for all s, since the continuous map $s\mapsto H(1,s)$ is a lifting of the constant mapping $s\mapsto h(1,s)$ and by uniqueness ought to be constant.

3.27. Proposition. Let $p: Y\to X$ be a covering, $x_0\in X$. If Y is arcwise connected, then the group $\pi_1(X,x_0)$ acts transitively on the fiber $p^{-1}(x_0)$.

Proof. Let $y\in p^{-1}(x_0)$ and $[\gamma]\in\pi_1(X,x_0)$. Define $y[\gamma]=\delta(1)$, where $\delta:[0,1]\to Y$ is the unique lift of γ with $\delta(0)=y$. Obviously $\delta(1)\in p^{-1}(x_0)$. Also, by 3.27, $\delta(1)$ does not depend on the choice of the curve $\gamma\in[\gamma]$. It follows from this definition that for any $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$ and $y \in p^{-1}(x_0)$:

$$(y[\gamma_1])[\gamma_2] = y[\gamma_1 * \gamma_2] = y([\gamma_1][\gamma_2]).$$

Finally for $y_1, y_2 \in p^{-1}(x_0)$ there is a $[\gamma] \in \pi_1(X, x_0)$ such that $y_1[\gamma] = y_2$. It suffices to consider a curve δ in Y going from y_1 to y_2. Then $\gamma = p\delta$ is a closed curve at x_0 and satisfies the above equality.

3.28. Theorem. Let $p: Y \to X$ be a covering, with Y arcwise connected and X simply connected. Then p is a homeomorphism.

Proof. Since $\pi_1(X, x_0)$ is trivial for every $x_0 \in X$, and since it acts transitively on $p^{-1}(x_0)$, we have $y_1[x_0] = y_2$ for every $y_1, y_2 \in p^{-1}(x_0)$, i.e. $y_1 = y_2$. Thus $p^{-1}(x_0)$ contains exactly one point, hence ~~it is~~ p is a homeomorphism by the definition of a covering.

3.29. Lemma. Let $p: Y \to X$ be a covering, $y_0 \in Y$, $x_0 = p(y_0)$. Then $p_\#: \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is a monomorphism.

Proof. Let $p_\#[\delta_1] = p_\#[\delta_2]$ for $[\delta_1], [\delta_2] \in \pi_1(Y, y_0)$. Then $[p\delta_1] = [p\delta_2]$ and hence by 3.26 we get $[\delta_1] = [\delta_2]$.

3.30. Proposition. Let $p: Y \to X$ be a covering and $y_0 \in Y$, $x_0 = p(y_0)$. Then $\pi_1(Y, y_0)$ is isomorphic to the subgroup H of $\pi_1(X, x_0)$ defined by $H = \{[\gamma] \in \pi_1(X, x_0) \mid y_0[\gamma] = y_0\}$.

Proof. If $[\gamma] = \operatorname{Im} p_\#$, then $[\gamma] = [p\delta]$ with $\delta \in \pi_1(Y, y_0)$. But then $y_0[\gamma] = \delta(1) = y_0$. Conversely, if $[\gamma] \in H$, then the lifting δ of γ with $\delta(0) = y_0$ satisfies $\delta(1) = y_0$, i.e. $[\delta] \in \pi_1(Y, y_0)$ and hence $[\gamma] \in \operatorname{Im} p_\#$.

3.31. Proposition. Let $p: Y_1 \to Y_2$, $p_2: Y_2 \to X$ be coverings

$$\begin{tikzcd} Y_1 \arrow[r, "p"] \arrow[dr, "p_1"'] & Y_2 \arrow[d, "p_2"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \pi_1(Y_1,*) \arrow[r, "p_\#"] \arrow[dr, "p_{1\#}"'] & \pi_1(Y_2,*) \arrow[d, "p_{2\#}"] \\ & \pi_1(X,*) \end{tikzcd}$$

and $*$ be the generic symbol for the "base point" in each space. Then $p_1 = p_2 \circ p$ is a covering and $\operatorname{Im} p_{1\#} \subset \operatorname{Im} p_{2\#}$.

Conversely, if $p_1: Y_1 \to X$ and $p_2: Y_2 \to X$ are coverings Y_1, Y_2 are arcwise connected and such that $\operatorname{Im} p_{1\#} \subset \operatorname{Im} p_{2\#}$, then there exists a unique covering $p: Y_1 \to Y_2$ with $p_1 = p_2 \circ p$.

false ~~Proof. (1) It is clear from the definition that p_1 is a covering.~~ The second statement follows from 3.21.

(2) Let $y_1 \in Y_1$ and δ_1 a curve from $*$ to y_1. Then $\gamma = p_1 \circ \delta_1$ is a curve in X originating in $*$ and it can be lifted uniquely to a curve δ_2 in Y_2, originating in $*$. Set $\delta_2(1) = p(y_1)$. To show that $p(y_1)$ is well defined, assume that δ_1' is another curve in Y_1 from $*$ to y_1. Then $[\delta_1 * \delta_1'^{-}] \in \pi_1(Y_1, *)$ and $p_{1\#}[\delta_1 * \delta_1'^{-}] \in p_{2\#}[\delta_1 * \delta_1'^{-}]$, hence $[p_1\delta_1 * p_1\delta_1'^{-}]$ leaves the base point of Y_2 invariant. In other words, if $\delta_2'^{-}$ is the unique lifting of $p_1\delta_1'^{-}$ originating in $\delta_2(1)$, then $(\delta_2 * \delta_2'^{-})(1) = * \in Y_2$. But this means that the inverse δ_2' of $\delta_2'^{-}$ goes from $*$ to $\delta_2(1)$, i.e. $\delta_2'(1) = \delta_2(1)$. It is clear that the mapping $p: Y_1' \to Y_2$ ~~and~~ satisfies $p_1 = p_2 \circ p$.

~~To show that p is a covering, let $y_2 \in Y_2$ and U be a distinguished neighborhood of $p_2(y_2)$ in X. Then $p_2^{-1}(U)$ is a neighborhood of y_2 and its component V_2 containing y_2 is simply connected. Consider $p_1^{-1}(U) \subset Y_1$.~~ Its components are homeomorphic to U, hence to V under $p_2^{-1} \circ p_1$.

To prove the continuity of p, let $y_1 \in Y_1$ and $y_2 = p(y_1) \in Y_2$. Consider a neighborhood V of y_2 homeomorphic to a distinguished neighborhood U of $p_2(y_2) = x$ in X (distinguished with respect to the covering p_2). By taking U smaller if necessary, we may assume that it is also distinguished with respect to p_1 and consider the component W of $p_1^{-1}(U)$ containing y_1.

Let $z \in W$ and $\zeta = \delta_1 * \alpha$, where δ_1 goes from $*$ to y_1 and α from y_1 to z.
~~will project onto $p_1 \circ \zeta$ from $*$ to $p_1(z) \in U$. Let ζ_2 be its unique lifting to Y_2. Then $p(z) = \zeta_2(1) \in p_2^{-1} p_1(z)$. If $p(z) \notin V$, then $p(z)$ would lie in a different component of $p_2^{-1}(U)$ than y_2~~

The existence of α is guaranteed by the fact that W is simply connected and hence arcwise connected. The curve ζ will project onto $p_1 \circ \zeta$ which goes from $*$ to $p_1(z) \in U$. Let ζ_2 be its unique lifting to Y_2 and δ_2 be the lifting of $p_1 \circ \delta_1$. By the uniqueness of the lifting, $\zeta_2(\frac{1}{2}) = \delta_2(1) = p(y_1) = y_2$. Therefore $\zeta_2(1) = p(z)$ lies in the same component of $p_2^{-1}(U)$ as y_2, i.e. in V.

Finally, p is covering. Consider a point $y_2 \in Y_2$ and a curve δ_2 from $*$ to y_2. Then $p_2 \circ \delta_2$ is a curve in X which admits a lifting to a curve δ_1 from $*$ to some point $y_1 \in Y_1$. Obviously $p(y_1) = y_2$ and hence $p^{-1}(y_2) \neq \emptyset$.
Let now V be the component of $p_2^{-1}(U)$ containing y_2, where U is a distinguished (for p_1 and p_2) neighborhood of $p_2(y_2)$ in X. Then V is simply connected.
~~However, let W be a component of $p^{-1}(V)$. Then $p_1|W = p_2 \circ p$ is a homeomorphism~~

~~onto its image, since~~ Moreover, $p_1(p^{-1}(V)) = p_2 p(p^{-1}(V)) = p_2(V) = U$, i.e. $p^{-1}(V) \subset p_1^{-1}(U)$. Therefore every component W of $p^{-1}(V)$ is contained in a component of $p_1^{-1}(U)$ and hence p_1/W is a homeomorphism onto its image in U. Consequently, for $z \in W$ we obtain $p(z) \in V$ and since $q_2 = p_2/V$ is a homeomorphism onto U, the equality $p_1(z) = q_2 p(z)$ yields $q_2^{-1} p_1(z) = p(z)$, whence p/W is a homeomorphism.

3.32. Theorem. Let $p_i : Y_i \to X$ $(i = 1,2)$ be coverings with Y_i arcwise connected and $\operatorname{Im} p_{1\#} = \operatorname{Im} p_{2\#} \subset \pi_1(X, *)$. Then there exists a homeomorphism $p : Y_1 \to Y_2$ such that $p_1 = p_2 \circ p$.

Proof. By 3.31 we get coverings p, p' such that the diagram is everywhere commutative.

$$\begin{tikzcd}
Y_1 \arrow[r, "p"] \arrow[dr, "p_1"'] & Y_2 \arrow[r, "p'"] \arrow[d, "p_2"] & Y_1 \arrow[r, "p"] \arrow[dl, "p_1"] & Y_2 \arrow[dll, "p_2"] \\
 & X & &
\end{tikzcd}$$

Let $y_1 \in Y_1$ and δ_1 be a curve going from $*$ to y_1. Then $p_1 \circ \delta_1$ is a curve in X and if δ_2 is its lifting to Y_2 originating in $*$, we have $p(y_1) = \delta_2(1)$. To get $p'p(y_1)$, we repeat the operation with δ_2. But then its projection on X is $p_1 \circ \delta_1$ and its lift to Y_1 is obviously δ_1. Therefore $p'p = id_{Y_1}$. A similar argument shows that $pp' = id_{Y_2}$.

3.33, Theorem. Let X be a locally simply connected locally? arcwise connected Hausdorff space; $\pi_1(X)$ the abstract group representing the fundamental group of X and H a subgroup of $\pi_1(X)$. There exists a unique covering $p: Y \to X$ of X by a locally simply connected arcwise connected Hausdorff space Y whose fundamental group is represented by H.

Proof. Choose a base point $*$ in X and define an equivalence relation on the set of all curves in X starting at $*$ as follows: consider the concrete realisation of $\pi_1(X)$ as $\pi_1(X, *)$ and denote again by H its given subgroup. Then two curves α_1 and α_2 starting at $*$ are equivalent, $\alpha_1 \sim \alpha_2$, if they have common endpoint and $[\alpha_1 * \alpha_2^-] \in H$ (observe that if $H = [*]$, then this means that α_1 and α_2 are homotopic).

Write $\{\alpha\}$ for the equivalence class of α under the relation $\sim$ and denote by Y the set of all these equivalence classes. Define $p: Y \to X$ by $p(\{\alpha\}) = \alpha(1)$.

Define a topology on Y as follows. For any $\{\alpha\} \in Y$ and any simply connected neighborhood U of $\alpha(1)$ in X define the U-neighborhood of $\{\alpha\}$ in Y as the set of all classes $\{\alpha * \beta\}$, where β is a curve in U, $\beta(0) = \alpha(1)$.

It is left as an exercise to show that Y and p have the required properties.

3.34, Definition. The unique simply connected covering $\tilde{X}$ of X, corresponding to $H = \{e\}$, is called the universal covering space of X.

3.35. Proposition. Let M^n be a connected C^r-manifold, $r \geq 1$, and $p: N \to M$ a covering of the topological space underlying M. There exists a standard C^r-structure on N of dimension n such that p is a C^r-mapping and p_* is an immersion.

Proof. Observe first that M^n is locally simply connected and loc. arcwise connected, therefore it admits a unique covering (up to a homeomorphism) by a locally simply connected arcwise connected Hausdorff space with given fundamental group H, where H is any subgroup of $\pi_1(M^n)$.

Consider now the covering $p: N \to M$ and let (U, φ) be a chart on M, where U is distinguished with respect to p. Let V be a component of $p^{-1}(U)$. Define $\psi: V \to \varphi(U) \subset \mathbb{R}^n$ by $\psi = \varphi p$. Then ψ is a homeomorphism. It is obvious that that every point $y \in N$ is contained in some such V. Moreover, if $V_1 \cap V_2 \neq \emptyset$ for V_1, V_2 with the above properties, then $\psi_2 \psi_1^{-1} = (\varphi_2 p)(\varphi_1 p)^{-1} = \varphi_2 \varphi_1^{-1}$ on $\psi_1(V_1 \cap V_2)$, which shows that $\psi_2 \psi_1^{-1}$ is a C^r-mapping. Thus the collection $\{(V, \psi)\}$ is a C^r-atlas. Moreover, for $z \in \psi(V)$, $\varphi p \psi^{-1}(z) = (\varphi p)(\varphi p)^{-1}(z) = z$, which shows that p is a C^r-mapping. We show now that $(p_*)_y: T_y N \to T_{p(y)} M$ is injective for every $y \in N$. Assume $p_*(\dot{\gamma}) = p_*(\dot{\delta})$, where $\dot{\gamma}, \dot{\delta} \in T_y N$. This means that if $\gamma \in \dot{\gamma}$, $\delta \in \dot{\delta}$ are C^r-curves through y in N, then $D(\varphi p \gamma)(0) = D(\varphi p \delta)(0)$ for some chart (U, φ) at $p(y)$. Obviously γ and δ lie in the same component of $p^{-1}(U)$ and U may be chosen distinguished. Therefore we get $D(\psi \gamma)(0) = D(\psi \delta)(0)$ which is equivalent to $\dot{\gamma} = \dot{\delta}$.

3.36. Remark. It is clear that 3.35 holds also for non-connected manifolds. Indeed, in this case the components of the manifold are open subsets, hence submanifolds and the above construction can be performed for each of them, yielding a non-connected covering of the entire manifold.

3.37. Remark. It is an immediate consequence of the inverse mapping theorem that the mapping p in 3.35 is a local C^r-diffeomorphism. However, the above proof shows directly that for every component V of $p^{-1}(U)$, where U is a distinguished chart, the mapping $p|V$ is a C^r-diffeomorphism onto U.

3.38. Proposition. Let G be a Lie group and e its identity element. The fundamental group $\pi_1(G,e)$ is abelian and its multiplication is induced by the multiplication in G.

Proof. We show that if $[\gamma],[\delta]\in\pi_1(G,e)$, then $[\gamma]*[\delta]=[\delta]*[\gamma]=[\gamma\delta]$, where $(\gamma\delta)(t)=\gamma(t)\delta(t)$ for all $0\le t\le 1$. Obviously $\gamma\delta$ is a closed curve at e if γ and δ are. Define now $h,g:[0,1]^2\longrightarrow G$ by

$$h(t,s)=\begin{cases}\gamma(2t-ts)\,\delta(st) & \text{for } 0\le t\le\frac12\\ \gamma(1-s(1-t))\,\delta(2t-1+s(1-t)) & \text{for } \frac12\le t\le 1\end{cases}$$

$$g(t,s)=\begin{cases}\gamma(st)\,\delta(2t-ts) & \text{for } 0\le t\le\frac12\\ \gamma(2t-1+s(1-t))\,\delta(1-s(1-t)) & \text{for } \frac12\le t\le 1\end{cases}$$

and verify that they are the required homotopies.

(Note that for every fixed s, $h(t,s)=\gamma(t')\delta(t'')$, with $t'=\tau_0+s(\tau_1-\tau_0)$, $(t''=\sigma_0+s(\sigma_1-\sigma_0))$, where $\tau_0,\tau_1,\sigma_0,\sigma_1$ are defined by the equalities $(\gamma*\delta)(t)=\gamma(\tau_0)\delta(\sigma_0)$, $(\gamma\delta)(t)=\gamma(\tau_1)\delta(\sigma_1)$).

3.39. Theorem. Let G be a connected Lie group. There exists a unique simply connected Lie group $\tilde G$ locally isomorphic to G. The manifold underlying $\tilde G$ is the universal covering manifold of the manifold G and the projection $p:\tilde G\longrightarrow G$ is a Lie homomorphism. Moreover, $\operatorname{Ker} p$ is algebraically isomorphic to $\pi_1(G,e)$ and is a central subgroup of $\tilde G$ (i.e. every element in $\operatorname{Ker} p$ commutes with every element in $\tilde G$).

Proof. The uniqueness of $\tilde{G}$ follows from 3.16. To prove existence, we introduce a multiplication on $\tilde{G}$. Recall that since $\tilde{G}$ is the universal covering space of G, its elements are homotopy classes of curves in G originating at e and with common endpoint. For $[\gamma] \in \tilde{G}$, we have $p[\gamma] = \gamma(1)$. Define now $[\gamma][\delta] = [\gamma\delta]$ for every $[\gamma], [\delta] \in \tilde{G}$ (note that $(\gamma\delta)(0) = e$, while $(\gamma\delta)(1) = \gamma(1)\delta(1)$). This multiplication is well defined, since homotopies $h, g : [0,1]^2 \to G$ between γ_1, γ_2 and δ_1, δ_2 respectively determine a homotopy $hg : [0,1]^2 \to G$ between $\gamma_1\delta_1$ and $\gamma_2\delta_2$. It is also obvious that the multiplication on $\tilde{G}$ defines a group structure with identity $[e]$ and inverse $[\gamma^{-1}]$, where $\gamma^{-1}(t) = [\gamma(t)]^{-1}$.

Let $[\gamma], [\delta] \in \tilde{G}$; then $p([\gamma][\delta]) = p[\gamma\delta] = (\gamma\delta)(1) = \gamma(1)\delta(1) = p[\gamma]\,p[\delta]$. Thus p is an algebraic homomorphism.

Moreover, $[\gamma][\delta] = q^{-1}(p[\gamma]\,p[\delta])$, where q denotes the restriction of p to a "distinguished" neighborhood of $[\gamma][\delta]$. Therefore the multiplication in $\tilde{G}$ is a C^r-mapping (with $r = \infty$ or $r = \omega$). Also $[\gamma]^{-1} = q^{-1}p([\gamma]^{-1}) = q^{-1}((p[\gamma])^{-1})$; hence inversion in $\tilde{G}$ is also a C^r-mapping. Consequently $\tilde{G}$ is a Lie group and p is a Lie homomorphism. Obviously p is also a local Lie isomorphism by 3.37.

Finally, if $[\gamma_0] \in \operatorname{Ker} p$ and $[\gamma] \in \tilde{G}$, then the mapping $[\gamma] \mapsto [\gamma][\gamma_0][\gamma^{-1}]$ of $\tilde{G}$ into $\operatorname{Ker} p$ (the latter is a normal subgroup of $\tilde{G}$) is continuous. But $\operatorname{Ker} p = p^{-1}(e)$ is a discrete subset of $\tilde{G}$ (by 3.23), hence this mapping is constant, i.e. $[\gamma][\gamma_0][\gamma^{-1}] = [e][\gamma_0][e] = [\gamma_0]$ or $[\gamma][\gamma_0] = [\gamma_0][\gamma]$. This shows that $\operatorname{Ker} p$ is a central subgroup.

Finally, $\operatorname{Ker} p$ and $\pi_1(G, e)$ are identical as sets, and by 3.38 they are isomorphic as groups (and even as "topological" groups, since $\operatorname{Ker} p$ is discrete).