Math 4500 Warmup #12, due 2/24/2017

Name: Student Number:

Read the Lecture Notes and Slides on the Exponential map. Review the differential of a map $F = (f_1, \ldots, f_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the differential at $x_0 \in \mathbb{R}^n$ is $DF = \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)$. There are two main theorems we need, the Chain Rule and the Inverse Mapping Theorem.

To compute $DF(x_0)(v)$, you can compute

$$\left.\frac{d}{dt}\right|_{t=0} F(x_0 + tv).$$

This is related to the directional derivative. $x_0 \in \mathbb{R}^n$, $DF(x_0)$ is a matrix and v is a column vector.

Theorem 0.1 (Chain Rule)**.** *If $F : \mathbb{R}^k \longrightarrow \mathbb{R}^m$ and $G : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, then $D(G \circ F) = DG \circ DF$.*

Theorem 0.2. *If $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable and such that $DF(x_0)$ is invertible, then there are neighborhoods $x_0 \in \mathcal{U} \subset \mathbb{R}^n$ and $F(x_0) \in \mathcal{V} \subset \mathbb{R}^n$ such that F is one-to-one on $\mathcal{U}$ and onto $\mathcal{V}$ with continuously differentiable inverse F^{-1} with differential at $y = F(x)$ given by $D(F^{-1})(y) = DF(x)^{-1}$.*

Remember the 1/2 hour rule.

Exercise I. Compute the first three terms of

$$e^X Y e^{-X}$$

in terms of Lie brackets. In other words take the products of X, Y where the powers add up to less than three, and rewrite in terms of brackets $[X, Y]$, $[X, [X, Y]]$, $[X, [Y, X]]$ and similar expressions.

Exercise II. Compute the differential of $F(X) = e^X$ at $X_0 = 0$.

Exercise III. Exercise 15(c) in Hall2015 asked you to show that $R = S(*)S^{-1}$ with $S \in SO(n)$. If you did not do this part, redo to receive missing credit.

It comes down to showing that if $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are two orthonormal bases, then the *Change of Variables* matrix is orthogonal, and one of the bases can be modified slightly so that the determinant becomes 1.

If you have trouble, ask.