Math 4500 Warmup #15, due 3/3/2017

Name: Student Number:

You can also find the notions below in the various Texts. The exercises look long, but they only test basic definitions.

Remember the 1/2 hour rule.

Exercise I. Let $\mathfrak{g}$ be a Lie subalgebra. The bracket is a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. So if we fix $X \in \mathfrak{g}$, and consider the bracket as a map in the second variable, we get a linear map denoted $\operatorname{ad} X : \mathfrak{g} \longrightarrow \mathfrak{g}$. Knowing the maps $\operatorname{ad} X$ is equivalent to knowing the bracket. The bracket satisfies the additional Jacobi identity. Check that this is eauivalent to

$$\operatorname{ad} X([Y, Z]) = [\operatorname{ad} X(Y), Z] + [Y, \operatorname{ad} X(Z)].$$

Write $gl(\mathfrak{g})$ for the linear transformations from $\mathfrak{g}$ to $\mathfrak{g}$. This forms a Lie algebra in the usual way, $[A, B] := AB - BA$. The formula above is interpreted as saying that the map $\operatorname{ad} X : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a derivation. Given a Lie algebra $\mathfrak{g}$, we can define a subspace of $gl(\mathfrak{g})$ called $Der(\mathfrak{g}) := \{D \in gl(\mathfrak{g})\ :\ D([X, Y]) = [DX, Y] + [X, DY]$.

Check that the linear subspace of derivations is a Lie subalgebra of $gl(\mathfrak{g})$.

Exercise II. Viewing $\operatorname{ad} X$ as a *function of* X, we get a linear map $\operatorname{ad} : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$.

Check that the Jacobi identity translates into $\operatorname{ad}[X_1, X_2] = [\operatorname{ad} X_1, \operatorname{ad} X_2] := \operatorname{ad} X_1 \circ \operatorname{ad} X_2 - \operatorname{ad} X_2 \circ \operatorname{ad} X_1$. This is interpreted as saying that $\operatorname{ad} : \mathfrak{g} \longrightarrow Der(\mathfrak{g})$ is a Lie homomorphism.

Conclusion: Every Lie algebra structure defines a Lie homomorphism $\mathfrak{g} \longrightarrow Der(\mathfrak{g}) \subset gl(\mathfrak{g})$ where the second one has the usual bracket structure.

Exercise III. Let $G \subset GL(n, \mathbb{R})$ be a matrix group with Lie algebra $\mathfrak{g} \subset M(n, \mathbb{R})$; we also write $M(n, \mathbb{R}) = gl(n, \mathbb{R})$ to emphasize it is the Lie algebra of $GL(n, \mathbb{R})$. For any element $g \in G$, we can define a continuous group homomorphism $A_g : G \longrightarrow G$ by the formula $A_g(x) := gxg^{-1}$.

1. Compute the differential dA_g. This means you must calculate $\frac{d}{dt}\big|_{t=0} A_g(e^{tX})$. Call the result $Adg : \mathfrak{g} \longrightarrow \mathfrak{g}$.
2. The answer should be $\operatorname{Ad} g(X) = gXg^{-1}$. Verify directly using this formula that $\operatorname{Ad} g : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Lie homomorphism. This means $\operatorname{Ad} g([X, Y]) = [\operatorname{Ad} g(X), \operatorname{Ad} g(Y)]$. Here X, Y are $n \times n$ matrices and the bracket is the usual $[A, B] = Ab - BA$. You must also verify that if $X \in \mathfrak{g}$, then for any $g \in G$, $gXg^{-1} \in \mathfrak{g}$ by using the definition of $\mathfrak{g}$.
3. Now consider the dependence in $g \in G$. Then $\operatorname{Ad} : G \longrightarrow GL(\mathfrak{g})$. Check that $\operatorname{Ad}(g_1 g_2) = \operatorname{Ad} g_1 \circ \operatorname{Ad} g_2$ and $\operatorname{Ad} I = I$. Be careful about the I on the left and the I on the right of the equation.
4. The previous exercise shows that $\operatorname{Ad} : G \longrightarrow GL(\mathfrak{g})$ is a group homomorphism. It is failry clear it is continuous. Compute its differential. This means $\frac{d}{dt}\big|_{t=0} \operatorname{Ad}(e^X)(Y)$. The answer should be $\operatorname{ad} X(Y)$.

If you have trouble, ask.