Chapter 8

Coordinates for Projective Planes

Math 4520, Fall 2017

8.1 The Affine plane

We now have several examples of fields, the reals, the complex numbers, the quaternions, and the finite fields. Given any field **F** we can construct the analogue of the Euclidean plane with its Cartesian coordinates. So a typical *point* in this plane is an ordered pair of elements of the field. A typical *line* is the following set.

$$L = \{(x, y) \mid Ax + By + Cz = 0\},$$

where A, B, C are constants, not all 0, in the field **F**. Note that we must be careful on which side to put the constants A, B, C in the definition of a line if our "field" is not a field but a non-commutative skew-field. Such a system of points and lines as defined above is called an *Affine plane*. Note that any two distinct points do determine a unique line.

On the other hand, suppose that we have a line L, and a line L'. Then we say that L and L' are *parallel* if L and L' have no points in common. There do exist parallel lines so an Affine plane is not a projective plane. We leave it as an exercise to show that being parallel is an equivalence relation on the set of lines in an Affine plane.

8.2 The extended Affine plane

Just as we constructed the extended Euclidean plane, we can construct the extended Affine plane. This is almost word-for-word the same definition as for the extended Euclidean plane. We repeat the definitions here:

A *point* is defined as a point in the Affine plane, called an *ordinary point*, or an equivalence class of parallel lines, called an *ideal point* or a *point at infinity*.

A *line* is defined as a line in the Affine plane, called an *ordinary line*, or a single extra line, called the *line at infinity*.

An ordinary point is *incident* to an ordinary line if it is a member of the set of points that define the Affine line. Each point at infinity is *incident* to the line at infinity, and it is *incident* to all of the lines in its equivalence class. These are the only incidences.

We leave it to the reader to check that all the axioms for a projective plane are satisfied by this extended Affine plane defined for any field or skew field **F**. The only case that should give any pause is when **F** is a skew field.

8.3 Affine planes — a different perspective

Although the extended Affine plane seems perfectly nice, it has a few drawbacks. It is not as democratic as it might be. All points seem to be created equally, but as with pigs in in the novel "Animal Farm", some points are more equal than others. And why is the line at infinity so special? This may not seem to be such a problem now, but later we really will want to "move" the ideal points to ordinary points and vise-versa. The extended Affine plane description will not be as convenient for this as the one we are about to define.

Recall that a line in the Affine plane is described by its three coordinates A, B, C. We regard the line as described by a row vector with three coordinates $[A, B, C]$. We must be careful though, because the same line has more than one description as such a vector. If t is a non-zero element of $\mathbf{F}$, then $[A, B, C]$ and $[tA, tB, tC]$ which we can write as $t[A, B, C]$) both describe the same line. (If $\mathbf{F}$ is a skew field we must be careful to only multiply on the left with t.) Furthermore, if a line L has two representations as $[A, B, C]$ and $[A', B', C']$ say, then there is a non-zero t such that $[A, B, C] = t[A', B', C']$.

We already regard a point in the Affine plane as a vector with two coordinates. But for notational convenience we regard the point as a column vector. Furthermore, we add a third coordinate, which is always 1. So we represent a point in the Affine plane as follows:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \tag{8.1}$$

where x and y are in the field (or skew field) $\mathbf{F}$. So a point given by (8.1) and a line given by $[A, B, C]$ are incident if and only if

$$[A, B, C] \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = Ax + By + Cz = 0,$$

where the above multiplication of vectors is regarded as multiplication of matrices. (This can also be thought of as an inner product of vectors, but we do not need any special properties of the real numbers such as sums of squares being positive, etc.)

With this notation what we have done is to identify the Affine plane as the plane $z = 1$ in a 3-dimendional vector space. A line in this space is regarded as the intersection of a plane through the origin (the plane $Ax + By + Cz = 0$) and the plane $z = 1$. The vector $[A, B, C]$ is perpendicular to the plane $Ax + By + Cz = 0$. See Figure 8.1.

8.4 Homogeneous coordinates

So far, all we have done is rewrite the notation for an Affine plane. We want to be able to extend the Affine plane to the projective plane using this notation. We use the 3-dimensional vector space $\mathbf{F}^3$. consisting of column vectors with 3 coordinates in the given field $\mathbf{F}$. We define our projective plane as follows.

A *point* is a line through the origin in $\mathbf{F}^3$. In other words, a point is all scalar multiples

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} t = \begin{pmatrix} xt \\ yt \\ zt \end{pmatrix},$$

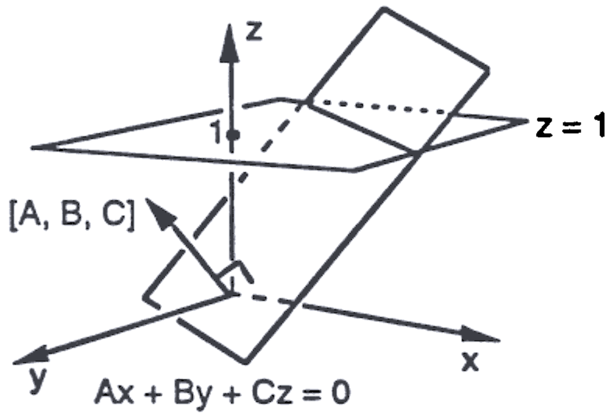

Figure 8.1

where t is an element of $\mathbf{F}$, and at least one of the coordinates x, y, z is non-zero.

A *line* is regarded as a plane through the origin in $\mathbf{F}^3$. More precisely, a line is the set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid [A, B, C] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ax + By + Cz = 0 \right\},$$

where A, B, C are constants, not all zero.

A point and a line are *incident* if the corresponding line in $\mathbf{F}^3$ is contained in the corresponding plane.

We leave it to the reader to check that this indeed satisfies the axioms of a projective plane.

We can use this point of view in the following way. We say that a projective point has homogeneous coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

when multiples of that vector determine the line in $\mathbf{F}^3$ that defines the projective point as above. So the same projective point usually has many homogeneous coordinates.

This notation has many advantages. For example, any point at infinity in our extended Affine description is given in homogeneous coordinates by

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$

where not both x and y are 0. The points at infinity are now treated democratically as all the others.

8.5 An example

Suppose that we take our projective plane to be the plane defined above coming from the smallest finite field $\mathbb{Z}_2$. Note that each line has only one non-zero point on it. There are $2^3 = 8$ vectors in $\mathbf{F}^3$ and thus 7 non-zero vectors in $\mathbf{F}^3$, each of which represents a different line. Figure 8.2 represents this finite projective plane, which is just another version of the Fano plane described in Chapter 2.

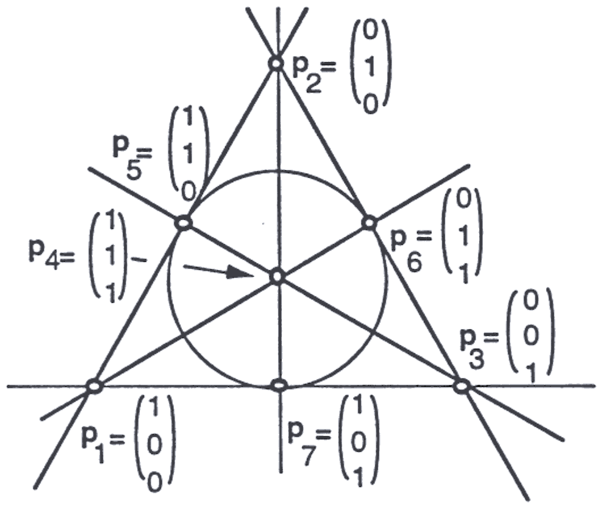

Figure 8.2

8.6 A projective plane where Desargues' Theorem is false.

Recall that in order to prove Desargues's Theorem in Chapter 4 we assumed that the projective plane could be extended to a projective three-space. Is that really needed?

We will now define a projective plane (due to F. R. Moulton in 1902) where Desargues' Theorem is false. It satisfies all three axioms of a projective plane, but there will be two point triangles which are in perspective with a point, but they will not be in perspective with respect to a line. So any proof of Desargues' Theorem would contradict the existence of Moulton's projective plane.

We will show the construction for an affine plane where the Desargues' property fails. This can be extended to a projective plane just as we did for the Extended Euclidean plane, by defining a point at infinity for each class of lines that do not intersect each other. Call this affine Moulton plane M.

Points in M are the same as points of the Euclidean plane.

Lines of M are defined as follows. For each m a real number or ∞, define the line L by the following equation:

$$\begin{aligned} x &= b && \text{if } m = \infty \\ y &= mx + b && \text{if } m \geq 0 \\ y &= mx + b && \text{if } m \leq 0, x \leq 0 \\ y &= 2mx + b && \text{if } m \leq 0, x \geq 0. \end{aligned}$$

The lines look like refracted light rays through glass or water, as in Figure 8.3.

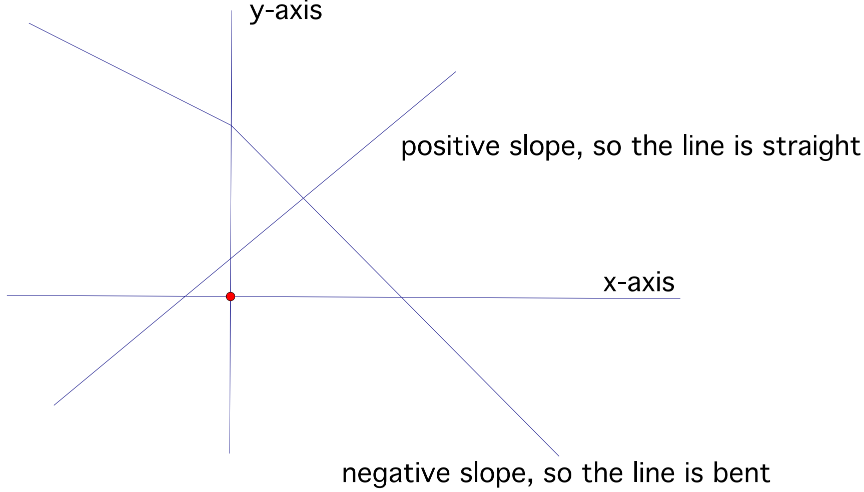

Figure 8.3

It is a nice exercise to show that this is in fact a projective plane. But now Figure 8.4 shows that the Desagues property fails for a judiciously chosen configuration of two triangles in perspective with respect to a point.

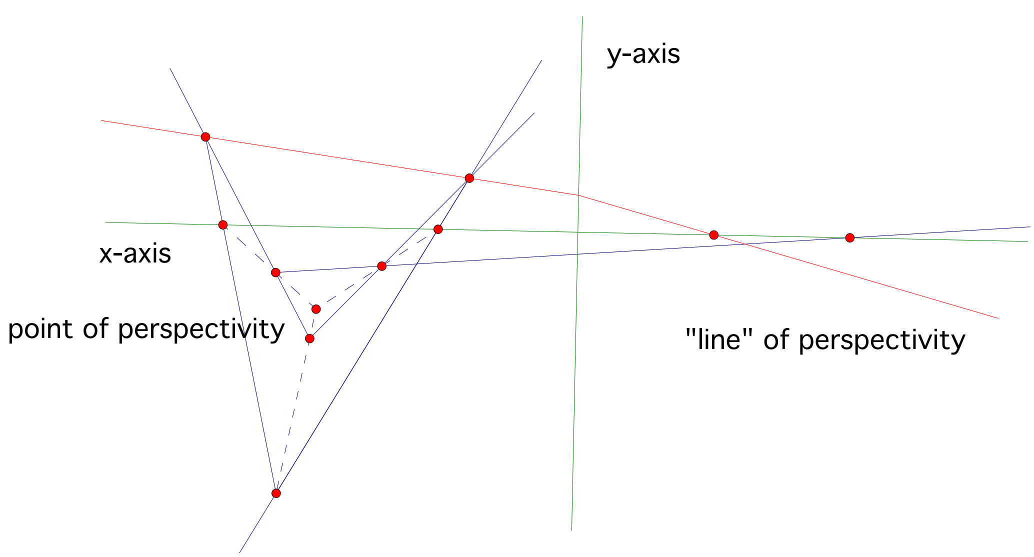

Figure 8.4

8.7 Exercises

1. Show that distinct points in an Affine plane, as defined in Section 8.1, lie in a unique Affine line, even when the underlying field is a skew field.

2. Show that the property of being parallel is an equivalence relation on the set of lines in an Affine plane, even when the field is a skew field.

3. Check that the axioms for a projective plane are satisfied for the extended Affine plane, even when the field is a skew field.

4. Check that the axioms for a projective plane are satisfied by the homogeneous coordinate description as in Section 8.4.

5. Suppose that the underlying field $\mathbf{F}$ is a finite field with q elements and we construct the projective plane using $\mathbf{F}$ with homogeneous coordinates. What is the order of this finite projective plane?

6. For each of the lines in Figure 8.2 find the coefficients $[A, B, C]$ that describe it.

7. Describe an axiom system for an Affine plane in the spirit of the axiom system for a projective plane without resorting to a coordinate field $\mathbf{F}$. Show that this Affine plane is the same as a projective plane with one line removed.

8. Show that the points and lines defined in Section 8.6 satisfy the axioms for a projective plane.

9. In class we showed that in general if a function $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ is a composition of projections in $\mathbb{RP}^2$ it may be necessary to use three projections since the composition of two projections from the original line to another and back again will necessarily have a fixed point, and there are such functions that have no fixed points. But the question is what happens if there is a fixed point. Consider the function $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ on the projective line where $f(-2) = 7, f(-3) = 5, f(3) = 2$.

   (a) Find a 2-by-2 matrix A such that it has the same values as f in homogeneous coordinates. For example,
   $$A \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$
   for some non-zero scalar α and similarly for $-3, 3$. (Hint: A can be taken to be an integer matrix with small integers.)

   (b) Show that the function f on $\mathbb{RP}^1$ has a fixed point. That is there is point x such that $f(x) = x$. (This is just a little bit of linear algebra. Think characteristic polynomial.)

   (c) Show that the function f can be written as the composition of just two projections using just one additional line in $\mathbb{RP}^2$. Show a picture how this can be done.