Math 4550 Midterm Exam-Solutions
Due Monday, October 15, 2018, Extended to Wednesday, October 17

In all the figures below solid heavy line segments are struts, dashed light line segments are cables, and medium line weight segments are bars.

1. Consider a triangulation of the unit sphere by congruent equilateral triangles, where there are n triangles at each vertex.

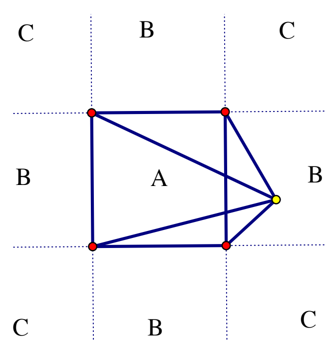
   (a) (5 points) Find the internal angle of each spherical triangle at each vertex.
   Solution: If the internal angle is θ, $n\theta = 2\pi$, so $\theta = 2\pi/n$.

   (b) (5 points) Find the area of each triangle as a function of n.
   Solution: The area of each triangle is $3\theta - \pi = 6\pi/n - \pi = (6/n - 1)\pi$.

   (c) (5 points) Find the number of triangles m in the triangulation as a function of n.
   Solution: $m(6/n - 1)\pi = 4\pi$, the area of the sphere. So $m = 4/(6/n - 1) = 4n/(6 - n)$.

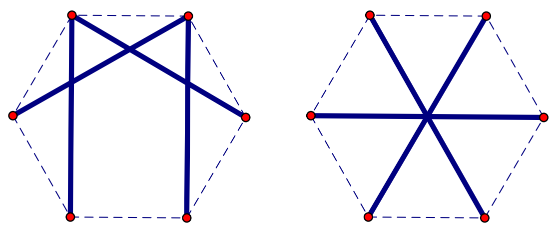
   (d) (5 points) For what values of n is m a positive integer, and what values correspond to the vertices of circumscribed regular polytopes.
   Solution: $m = 20, 8, 4, 2$, for $n = 5, 4, 3, 2$, respectively, corresponding to the regular icosahedron, octahedron, tetrahedron, respectively. The last case, $n = 2, m = 2$ does not correspond to a regular polytope exactly, but to a hemisphere with 3 points on its boundary.

2. (20 points) In one of the regions A, B, C in the plane place a vertex in the interior and join it with bars to the vertices of the square of bars as shown. In each case determine when the resulting (bar) framework is universally rigid. (Hint: You only need to consider rigidity in dimension 3, and one case is an old friend. You can also experiment with folded paper.)

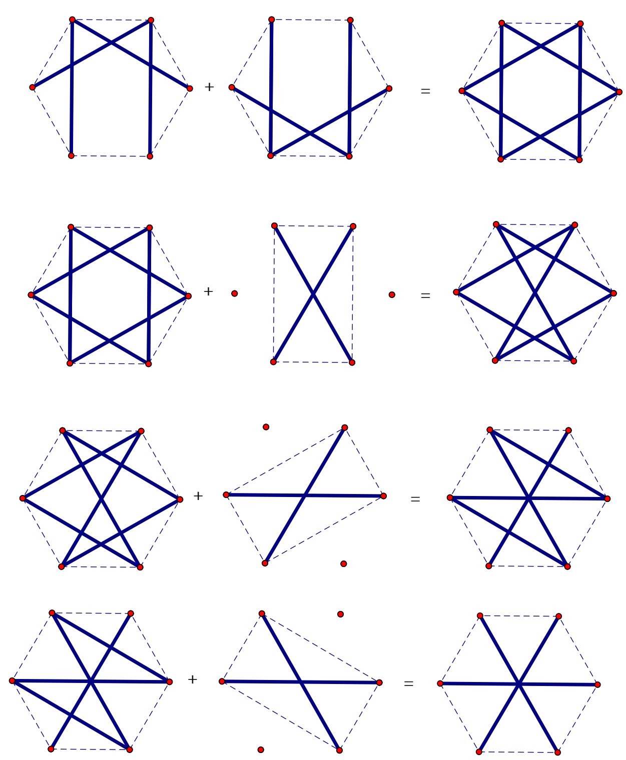


Solution: When the yellow vertex is in the region B, the configuration is a Cauchy Polygon, and thus is universally rigid. When it is in region A or C, the whole bar framework flexes non-trivially in $\mathbb{R}^3$. For example when the yellow vertex is in region A, the opposite vertices of the red square can move together, while the other pair of red vertices can be placed to extend the motion. If the yellow vertex is placed in the center of the red square, it is easy to see the motion.

3. (20 points) We proved that the planar tensegrity on the left below, a Cauchy polygon, is super stable. The vertices in both cases are those of a regular hexagon. Show that the tensegrity on the right is also super stable. Please don't fuss with the stress matrix explicitly. Do this by drawing pictures and superimposing the stresses from other tensegrities and using symmetry. For example we know that any Cauchy polygon, with with 4 or more vertices forming a convex polygon is super stable. For extra credit, if we do a small perturbation on the the vertices of the right tensegrity, is it always still super stable?



Solution: The following shows how to add a sequence of tensegrities, with a PSD stress to the left one to get the one on the right. At each stage scale the posiitve cable stresses to cancel with the negative strut stresses, using the symmetry. "Most" small perturbations of the right tesegrity will not have any equilibrium stress.



4. The Figure below, the vertices $2, 3, 4, 5$ form a square of side length 1, and the vertices $1, 2, 5$ form and equilateral triangle, with vertex 6 in its center.

   (a) (5 points) Calculate the stress matrix Ω when the stress $\omega_{2,3} = 1$.

   Solution: You add the stress matrix for the cable square with strut diagonals and the stress matrix for the regular triangle with boundary as struts and cables connecting the center to the vertices, cancelling the strut and cable stresses on the $2, 5$ members. You get the following:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & -3 \\ 1 & 2 & -1 & 1 & 0 & -3 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 2 & -3 \\ -3 & -3 & 0 & 0 & -3 & 9 \end{pmatrix}.$$

   (Remember the sign change going from the stress ω_{ij} to the i, j coordinate in Ω, for $i \neq j$.).

   (b) (5 points) Is this tensegrity super stable? Why or why not?

   Solution: No, it flexes into $\mathbb{R}^3$.

   (c) (5 points) Is Ω positive semi-definite? Explain.

   Solution: Yes, it is the sum of two PSD matrices.

   (d) (5 points) Is there any other configuration satisfying the cable and strut conditions such that the distance between vertex 2 and vertex 5 is not 1? Explain.

   Solution: No, the left hand implies that the distance between the $2, 5$ vertices cannot decrease. The right-hand triangle with its center implies that the $2, 5$ vertices cannot increase.

   (e) What is the rank of Ω? (5 points for any calculation, 10 points without calculating the rank of Ω numerically.)

   Solution: The rank of Ω is 2, because the largest dimension that its vertices can span is 3-dimensional because the rank of its (co-)kernel is 4. $6 - 4 = 2$. The square vertices span a 2-dimensional space, and the triangle spans another 2-dimensional space, and these 2 2-dimensional spaces intersect along the 1-dimensional space through the $1, 2$ points.