Math 4550 Homework #7, Solutions

Problems due in class Friday, November 3: Read Section 3.2 in my book. Extra credit if you build some interesting tensegrities.

Consider the planar a-by-b bar grid of squares in the plane, as below:

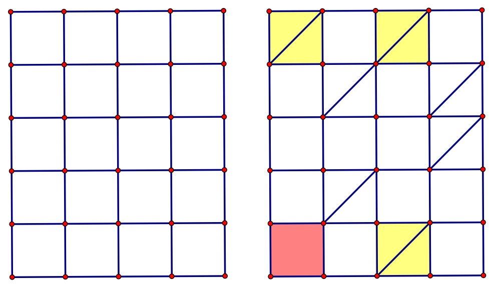


Figure 1: This is a bar framework of $a = 5$ rows of squares, and $b = 4$ columns of squares. The framework on the right has $c = 8$ of the squares braced with diagonal bars.

1. For an a-by-b bar grid of squares in the plane with c braced squares, calculate n the number of vertices and m the number of bars in the final braced framework. What is the minimum value for c, in order for the braced framework to be infinitesimally rigid?

   The number of vertices is $n = (a + 1)(b + 1)$, and the number of edges (bars) is $m = a(b+1)+(a+1)b+c = 2ab+a+b+c$. To be infinitesimally rigid, we must have $2ab+a+b+c = m \geq 2n - 3 = 2(a + 1)(b + 1) - 3 = 2ab + 2a + 2b - 1$, or $c \geq a + b - 1$.

2. Can the framework on the right in Figure 1 be infinitesimally rigid in the plane? Why?

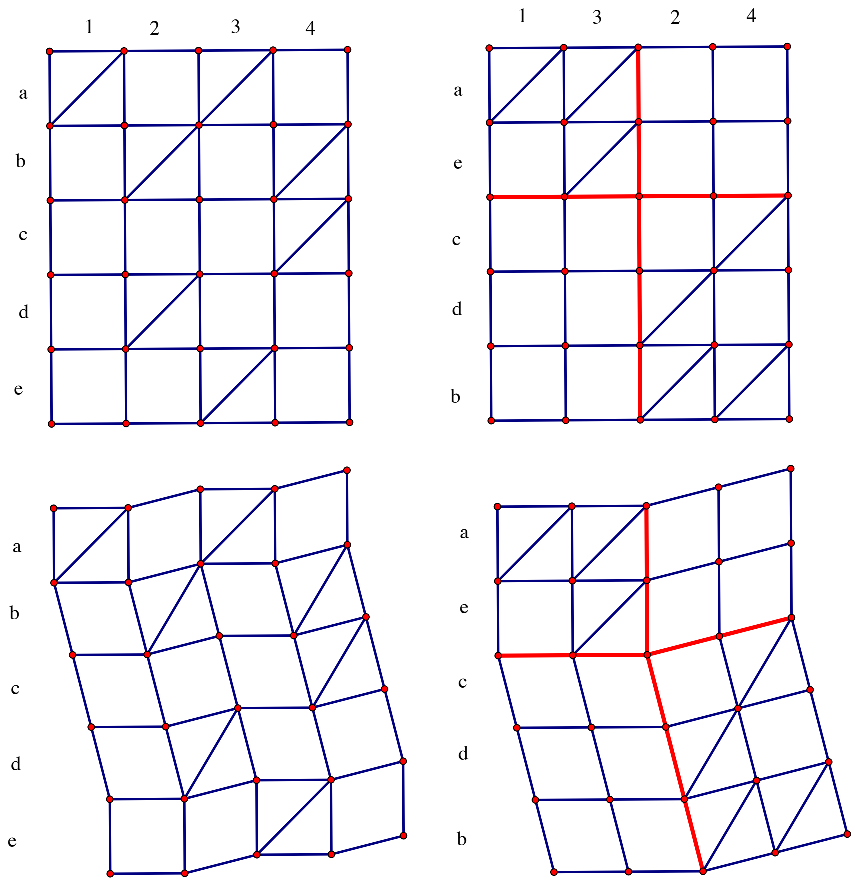
   No, because $c = 7 < a + b - 1 = 8$.

3. Show that if 3 corners of a rectangle in the grid are braced so is the fourth. For example, in the grid on the right, the three yellow squares are braced, so the red square is also braced.

   Since all the squares in the structure deform to rhombi, opposite sides remain parallel. So all the bottom and top edges of small squares of any column deform to parallel line segments. Similarly all the left and right edges of small squares in a row deform to parallel line segments. When one square is braced, all bottom/top edges in that column remain perpendicular to left/right edges in that row. When squares are braced as in the figure, as you walk around the yellow braced squares, the up/down edges of the red square remains perpendicular to its left/right edges, and thus is effectively braced.

4. If a braced grid framework is rigid, so is the grid you get by permuting the rows or permuting the columns. Permute the rows and permute the columns in the braced grid on the right in Figure 1 so that there is a horizontal line that separates two sets of braced squares, and similarly there is vertical line that separates the same two sets of squares. Use that to find a non-trivial flex of the whole framework. For extra credit, find a corresponding flex of the given framework on the right.

   The following figure shows the way to proceed with the flexing.



5. If you have a braced grid as above, consider the graph with two sets of vertices, one corresponding the rows of squares and another corresponding to the columns of squares. Place an edge between the row vertex and the column vertex if the corresponding square is braced. What is the relation between the connectivity of that graph and the rigidity of the braced framework?

   If you continue to fill in the squares, as in Figure 1, and all the squares are braced, then the grid is rigid. But in terms of the bipartite graph, the filling in process completes any three edges in sequence to a cycle of four. If this is continued then we will eventually connect all the vertices of one partition to all the vertices of another partition. If either of those partitions are not the whole graph, then the process as above shows how to flex the whole grid. Otherwise the whole grid is rigid and the bipartite graph is connected.

6. Call G *Laman graph* if $m = 2n - 3$, where n is the number of vertices of G, m is the number of edges of G, and G is generically rigid in $\mathbb{R}^2$. Suppose that all the vertices of a Laman graph G have degree 3, i.e. each vertex is adjacent to exactly 3 edges. What is the number of vertices in such a Laman graph?

   If all the vertices of the Laman graph have degree 3, then $3n = 2m$, since each edge is adjacent to two edges and each vertex is adjacent to 3 edges. Since $m = 2n - 3$, $3n = 2(2n - 3)$, which implies $n = 6$, a minimum and a maximum for the conditions.

7. Show that the following are Laman graphs by doing a Hennenberg operation, or by adding a vertex of degree 2 one at a time, starting with a triangle.

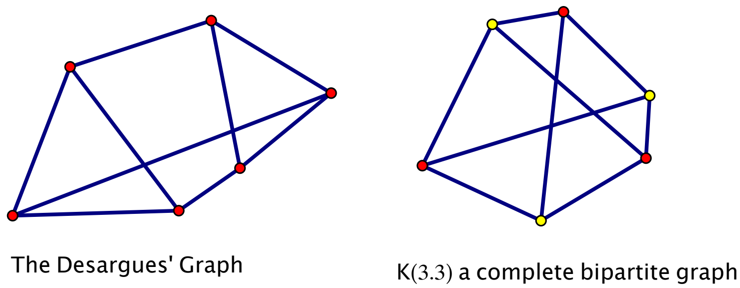
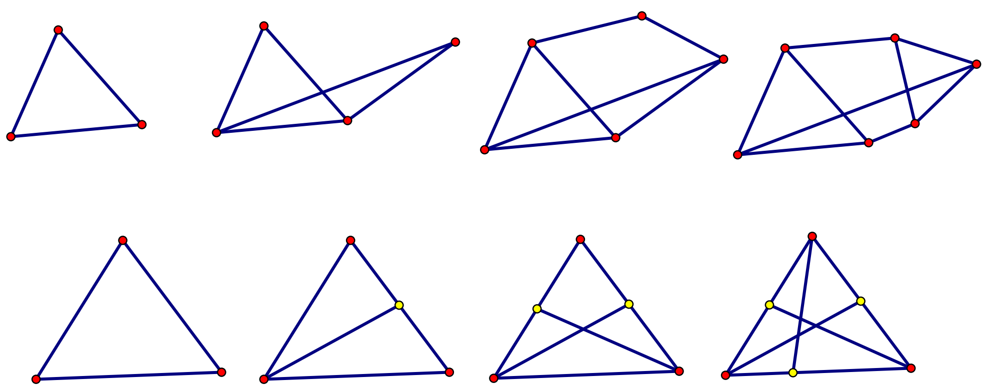


Figure 2

The following figure shows the way to construct the graphs with Henneberg moves and attaching vertices of degree 2.



8. In the proof of Laman's Theorem, we started with a vertex of degree 3, removed it, and joined one pair of the adjacent vertices to get a graph with one fewer vertices. Find a Laman graph and a degree 3 vertex, such that when it is removed any pair of the adjacent vertices can be joined with a bar to get a smaller Laman graph.

   The $K(3,3)$ graph has such a property. We showed that it is a Laman graph in Problem 7. When you remove a vertex and all its adjacent edges, you get the complete bipartite graph $K(2,3)$, where the vertices adjacent to the removed vertex are the 3 vertices in the other partition. There is a graph isomorphism that interchanges any pair of those 3 vertices. Since $K(3,3)$ is Laman and we know from the proof of Laman's Theorem one pair of those vertices can be joined to create a smaller Laman graph, so any 2 of the 3 pairs can be joined to create a Laman graph.