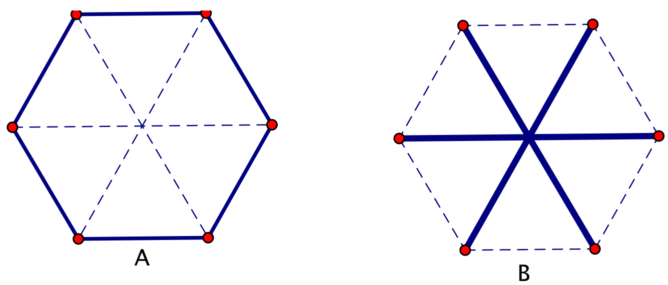
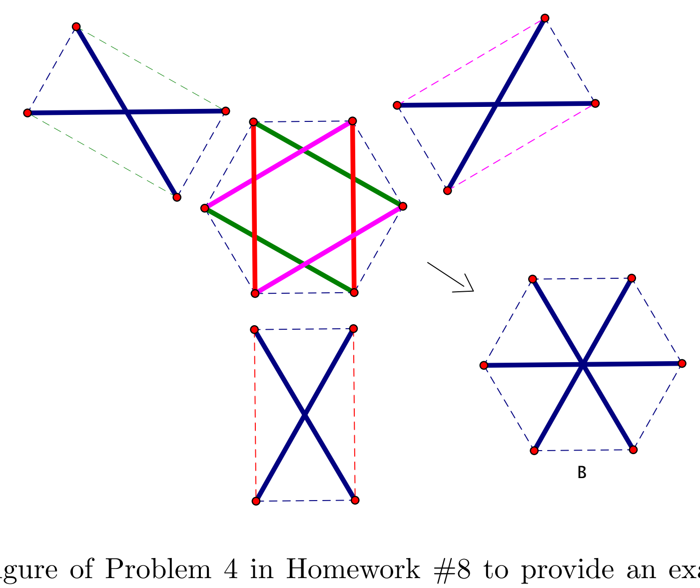
Math 4550 Homework #10-solutions

1. The Figure A below, a bar and cable tensegrity in the plane, is not rigid in the plane. The idea is to prove that with absolutely NO numerical calculations. (Five points will be taken off if any such calculations are used.) Just draw pictures and appeal to the results in class.



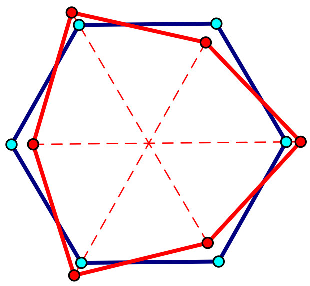
   (a) Show that the hexagon in Figure B with cables on its edges and struts for its long diagonals is super stable. You can prove that a convex quadrilateral with cables on its boundary and struts for its diagonals is super stable, and add appropriate such quadrilaterals to show the super stability of Figure B using the symmetry. You can use the three criteria to show super stability.

   The four tensegrities on the left are all superstable, while the one in the middle is the symmetrized Cauchy polygon (also the sum of super stable quadrilaterals). The Cauchy polygon is, by itself, super stable on all the vertices, and so the addition of other PSD terms will still create a super stable configuration, while the colored struts and cables are scaled to cancel.



   (b) Use the figure of Problem 4 in Homework #8 to provide an example that is a flex of the tensegrity in Figure A. Use the result in Part (a) to show that the bar and cable conditions are preserved.

   The following shows a figure of flexed 3-fold symmetric bar polygon, starting from Figure A. The symmetry insures that it exists. The long diagonals inside are all the same length (by the symmetry) and so must all either increase in length, stay the same length, or decrease strictly. The first two cases cannot happen, since that would contradict the universal rigidity of Figure B. So the diagonal dashed members must decrease in length as cables. Note that the configuration below comes from the push-me-pull-you process, but you need some argument to see that length of the diagonals decrease during the flex.



2. (a) The following Figure shows a 3-dimensional half-octahedral tetrahedral truss that is commonly used for supporting roofs. Prove that it is generically rigid. Hint: You can use the Henneberg construction in Section 3.2.4 in my book.

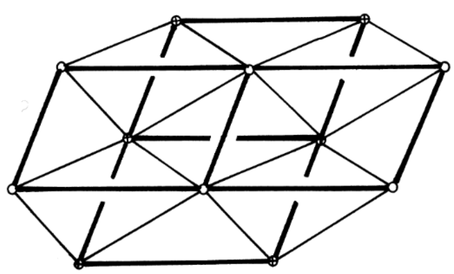
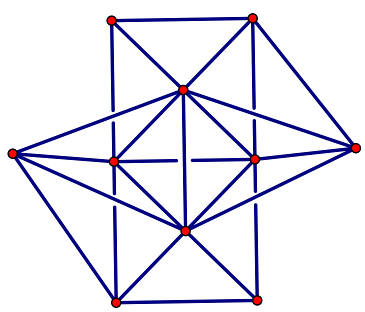
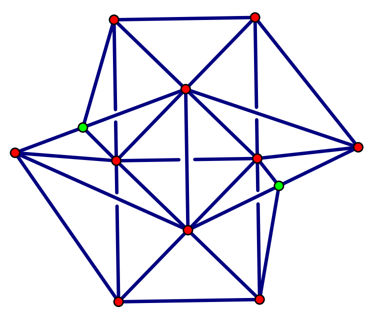


Figure 1

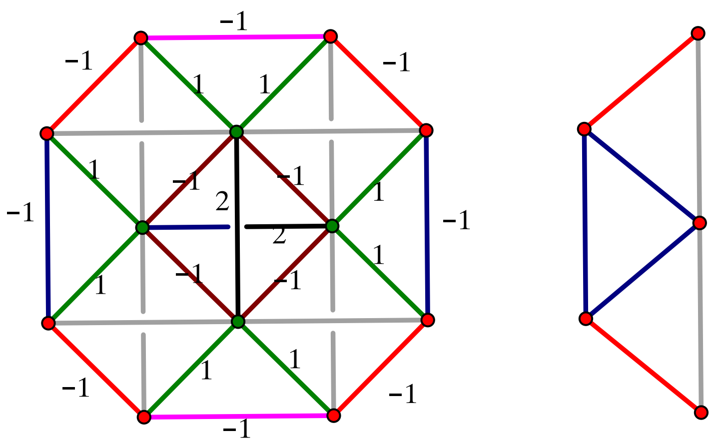
The figure on the left starts with the middle tetrahedron and adds degree 3 vertices cyclicly as shown. Then two Henneberg moves are performed at the blue points to get the graph of the octet truss.



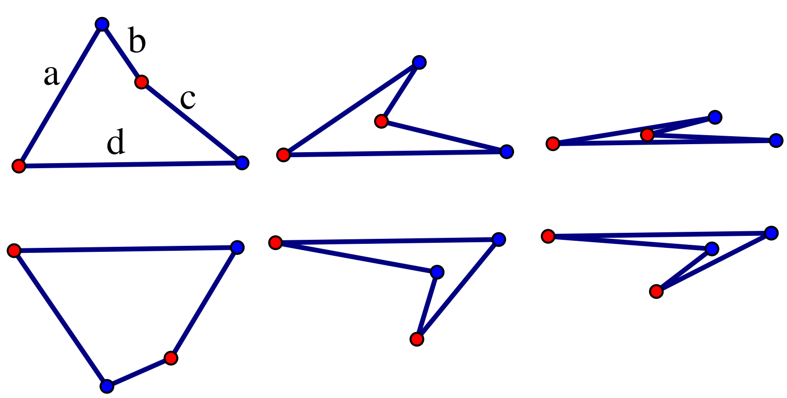


(b) Show that the half-octahedral tetrahedral truss above is not infinitesimally rigid in $\mathbb{R}^3$. Hint: Look for a non-zero equilibrium stress.

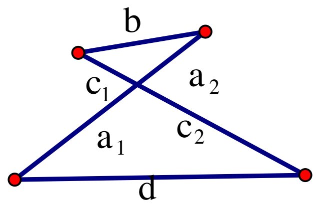
(We assume that all the bars have the same length.) The figure shows a top and side view of the half-octahedral tetrahedral truss. The grey edges have 0 stress because from the side view, all the colored edges are coplanar and the grey edge is not in their plane. The outside edges all have the same stress, -1 and the green edges have stress $+1$ providing equilibrium at the red vertices in the top view. The dark brown edges on the internal tetrahedron are -1 again to to maintain equilbrium the z direction in the top view. The black vertex stress is 2 to maintain equilibrium in the x, y direction in the top view. Since the bar framework is generically isostatic, any non-zero equilibrium stress implies that the framework is not infinitesimally rigid.



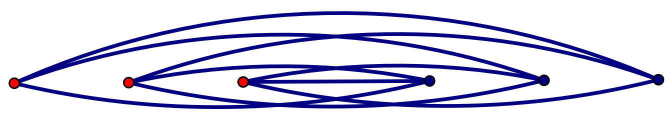
3. (a) Suppose you have a quadrilateral in the plane whose bar lengths, in order, are a, b, c, d as shown, and $a - b + c - d = 0$. Show that either the quadrilateral lies on a line, or it is embedded as in the Figure below. Hint: If some pair of edges cross, use the triangle inequality to get a contradiction.



If there is a crossing as in the figure below, suppose segment a intersects segment c, and decompose them as shown from their intersection, where $a = a_1 + a_2$, and $c = c_1 + c_2$. Then by the triangle inequality, $a_1 + c_2 \geq d$, and $c_1 + a_2 \geq b$, where, unless all the points are in a line, at least one of those inequalities is strict. So $a + c = a_1 + c_2 + c_1 + a_2 > b + d = a + c$ a contradiction.



(b) Consider a realization of the graph $K(3,3)$ in the line as in the Figure below, where one partition of the vertices, say the blue vertices, are separated along the line from the other partition, the red vertices. Show that this bar framework is globally rigid in the plane. Hint: Use Part a) and the theorem that says that the graph $K(3,3)$ does not have a topological embedding in the plane.



By part (a) there can be no crossing of the quadrilateral. Since $K(3,3)$ does not have any embedding in the plane, it must lie in the line. We have assumed (implicitly) that the four points in the initial realization must be distinct. If there is another realization in the line then they must satisfy an equation of the sort $a \pm c = \pm c \pm d$, eight possibilities. Each case other than $a + c = b + d$ implies some pair of points overlap or one of the edge lengths is 0. For example, if $a - c = b + d$, then $c = 0$, or if $a - c = b - d$, then $a = b$ and $c = d$, etc.