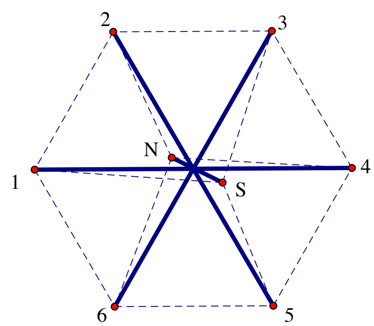
Math 4550 Homework #9

Problems due in class Friday, November 16: Read Section 5.9 in my book. Extra credit if you build some interesting examples.

1. The figure shows a regular hexagon, where struts and cables are indicated, The N and S vertices are on top of each other with a strut between them. All 12 cables have stress 1, and the 4 struts have stress -1. Calculate the stress matrix and its rank. In particular, find a universal configuration for the indicated stress. That is, find the maximum dimension of the affine span of a configuration with the given stress. Is the given tensegrity, in the plane, universally rigid?



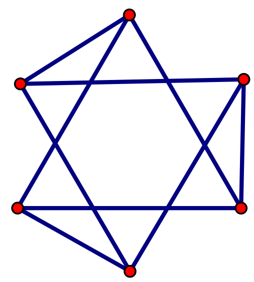
When the N and S vertices are counted as 9 and 10, you get:

$$\Omega = \begin{pmatrix} 2 & -1 & 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 2 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

The rank of Ω is 4, and so this configuration is the projection of another configuration in $\mathbb{R}^3$, which is its universal configuration, whose maximal affine span is 3-dimensional. The planar configuration above is universally rigid, since it has another PSD stress on the hexagon determined the first 6 vertices, and N, S are attached rigidly to those 6 vertices in the plane.

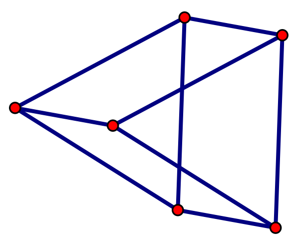
The configuration in $\mathbb{R}^3$ can be taken to be the unit cube with cables on its edges and struts for its four long diagonals, where the figure above is the orthogonal projection parallel to one of those long diagonals. (You can see this by looking at the picture above with some imagination.) By adding a square tensegrity in each face of the cube and rectangular tensegrities along opposite parallel sides appropriately weighted, we see that Ω is PSD of rank 4.

2. The following bar framework has 3-fold rotational symmetry about the center of the congruent equilateral triangles. Is that bar framework rigid in the plane? Is every realization of this framework with the same bar lengths rigid in the plane?



The given framework is infinitesimally rigid in the plane, and thus locally rigid. This framework is generically rigid in the plane (from other problems) and is Laman. So it is infinitesimally rigid if and only if it has no equilibrium stress. By tracking the signs of a non-zero equilibrium stress at each vertex, we see there can be only the 0 stress.

The following is another equivalent realization of the same graph that is flexible.



3. Suppose that two generically realized isostatic rigid bar frameworks G_1 and G_2 are joined together generically in the plane at two vertices a and b, where there is no bar connecting a and b in either graph.

   (a) Show that $G_1 \cup G_2$ is not isostatic, since it has too many ~~vertices~~ bars.

   If G_i has n_i vertices and m_i bars, $i = 1, 2$, then $m_i = 2n_i - 3$, and the number of vertices in $G_1 \cup G_2$ is $n_1 + n_2 - 2$, while the number of bars in $G_1 \cup G_2$ in $m_1 + m_2$. But $2(n_1 + n_2 - 2) - 3 = 2(n_1 + n_2) - 7$, while $m_1 + m_2 = 2n_1 - 3 + 2n_2 - 3 = 2(n_1 + n_2) - 6$, one more than what is needed for generic rigidity.

   (b) Show that there is a bar that can be removed such that $G_1 \cup G_2$ is isostatic.

   Clearly $G_1 \cup G_2$ in generically rigid, and so there must be a bar that can be removed and maintain generic rigidity.