Lecture 12. Basic properties of the Fourier transform

1 Smoothness and decreasing

Fourier coefficients of a finite function of class $C^m$ decrease as $|k|^{-m}$. The Fourier transform has a similar property. This was proved in the previous lecture. Here we prove a more general statement.

**Theorem 1** Let the Fourier transform of a function $f^{(m)}$ be bounded:

$$|\mathcal{F}(f^{(m)})| < C.$$ 

Then for any $|\alpha| \geq 1$,

$$|\hat{f}(\alpha)| < \frac{C}{|\alpha|^m}.$$  (1)

**Proof** The proof goes by induction in $m$.

Base of induction: $m = 1$. Let

$$\mathcal{F}(f')(\alpha) = \int f'(x)e^{-i\alpha x}dx = \int e^{-i\alpha x}df(x) = i\alpha \int f(x)e^{-i\alpha x}dx = i\alpha \mathcal{F}(f)(\alpha).$$

Then,

$$|\hat{f}(\alpha)| = \frac{|\hat{f}'(\alpha)|}{|\alpha|}.$$ 

The base is over.

The induction step consists of the application of this inequality and implies (1). \qed

2 Decreasing and smoothness

An inverse theorem is true: if the Fourier transform of some function belongs to $L_2(\mathbb{R})$ and decreases as $c|\alpha|^{-m}$, then the function itself is $m$ times differentiable. We will not prove this theorem. Instead, we will prove a similar and more important fact.
3 A space of rapidly decreasing functions

Definition 1 A function defined on \( \mathbb{R} \) is called rapidly decreasing if it is infinitely smooth and decreases at infinity faster than any power of \( |x| \). The space of all such functions is denoted by \( S \).

Theorem 2 The Fourier transform maps the space of rapidly decreasing functions onto itself.

Proof Consider an arbitrary function \( f \in S \). It belongs to \( L^2(\mathbb{R}) \) (prove it!). It is infinitely smooth. Hence, its Fourier transform decreases faster than any power at infinity. Let us prove that for any function \( f \in S \) its Fourier transform is infinitely smooth. Indeed, the function \( f \) decreases faster than any power at infinity. The same property has a function \( x^m f(x) \) for any natural \( m \). Hence, all the functions

\[
g_k = (ix)^k f(x)e^{-ix\alpha}
\]

for any \( \alpha \in \mathbb{R} \) are majorized by a function \( \frac{C}{1+x^2} \). The integral of this function over \( \mathbb{R} \) converges. Hence, the integral

\[
I_k(\alpha) = \int (ix)^k f(x)e^{ix\alpha}dx
\]

admits the derivation of the integrand in \( \alpha \). This implies:

\[
I_{k+1}(\alpha) = I_k'(\alpha).
\]

Together with the relation \( I_0 = \hat{f} \), this proves the theorem. \( \square \)

4 The convolution

An analog of the convolution occurred in the study of \( \delta \)-sequences. Namely, we have proven that if \( \Delta_n \) is a \( \delta \)-sequence, and \( f \) is a finite continuous function, then

\[
\int f(x)\Delta_n(x-y)dx \Rightarrow f(y).
\]  

(2)

Remark 1 In this theorem finiteness may be replaced by boundedness.

The operation that brings a pair of functions \( f, \Delta_n \) to the left hand side of the relation (2) is non-symmetric. Its symmetric analog is defined as follows.
**Definition 2** A convolution of two functions on the line (denoted by \( f \ast g \)) is a function

\[
f \ast g : y \mapsto \int f(x)g(y - x)dx.
\]

An equivalent definition follows.

**Definition 3** Consider a differential 1-form in the plane:

\[
\omega_x = f(x)g(y)dx
\]

Let:

\[
f \ast g(z) = \int_{l_z} \omega_x.
\]  

(3)

Here \( l_z \) is the line \( x + y = z \), whose orientation is inherited from that of the \( x \)-axis via the projection \((x, y) \mapsto x\).

We suppose that all the integrals mentioned in this definition converge.

**Theorem 3** The convolution is symmetric:

\[
f \ast g = g \ast f
\]

**Proof** By definition,

\[
g \ast f = \int_{l_z} g(x)f(y)dx := \int_{l_z} \omega'_x
\]

The symmetry \( s : (x, y) \mapsto (y, x) \) brings the form \( \omega'_x \) to the form \( \omega_y = f(x)g(y)dy \). Note that on \( l_z : \omega_x + \omega_y = 0 \). The symmetry \( s \) brings \( l_z \) to \(-l_z\). Hence

\[
\int_{l_z} \omega'_x = \int_{-l_z} s^* \omega'_x = \int_{-l_z} \omega_y = -\int_{-l_z} \omega_x = \int_{l_z} \omega_x.
\]

\( \square \)

**Theorem 4** (Associativity of convolution) The convolution is associative:

\[
(f \ast g) \ast h = f \ast (g \ast h)
\]

**Proof** We give only a sketch here. Both convolutions in the theorem above equal to the “triple convolution”

\[
f \ast g \ast h(u) = \int_{L_u} f(x)g(y)h(z)dx \wedge dy,
\]

(4)

where \( L_u \) is a plane \( x + y + z = u \) that inherits the natural orientation of the \( x, y \) plane via the projection \((x, y, z) \mapsto (x, y)\).

\( \square \)
5 Convolution and the Fourier transform

Theorem 5 The Fourier transform brings the convolution to the product, and the product to the convolution (the latter with the coefficient $\frac{1}{2\pi}$).

Proof Let us prove that
\[ F(f \ast g) = \tilde{f} \cdot \tilde{g} \] (5)

By (3),
\[ F(f \ast g)(\alpha) = \int_R \left( \int_{l_x} f(x)g(y)dx \right) e^{-i\alpha z}dz = \int_{R^2} e^{-i\alpha(x+y)}f(x)g(y)dxdy = \]
\[ \int_R e^{-i\alpha x}f(x)e^{-i\alpha y}g(y)dxdy = \tilde{f}(\alpha)\tilde{g}(\alpha). \]

This proves (5). The equality
\[ F(fg) = \frac{1}{2\pi} \tilde{f} \ast \tilde{g} \] (6)

follows from (5) and the formula for the iterated square of the Fourier transform : $F^2 = 2\pi S$. Here $S$ the “reversion” operator:

\[ Sf : x \mapsto f(-x). \]

Let us apply the Fourier transform to both parts of the equation (5). We get
\[ F^2(f \ast g) = S(f \ast g) = 2\pi F(\tilde{f} \cdot \tilde{g}). \]

Take now $\tilde{f}$ and $\tilde{g}$ as the original functions and denote them by $\varphi$ and $\psi$. Then $\tilde{f} = \frac{1}{2\pi}S\varphi, \ g = \frac{1}{2\pi}S\psi$, and we get:
\[ \frac{1}{2\pi}S(SF\psi \ast SF\varphi) = F(\varphi \cdot \psi). \] (7)

It remains to prove that the left hand side equals $\frac{1}{2\pi}SF\psi \ast SF\varphi$. Indeed, let $u$ and $v$ be arbitrary functions from $S$; then

\[ S(Su \ast Sv) = u \ast v. \] (8)

Equalities (7) and (8) imply (6). Let us prove (8):
\[ S(Su \ast Sv)(z) = (Su \ast Sv)(-z) = \int_{-z} u(-x)v(-y)dx = -\int_{-z} u(x)v(y)dx = (u \ast v)(z). \]

\[ \square \]
6 Fourier transform and δ-sequences

It appears that δ-sequences are closely related to Fourier transforms of stretched function. Namely, the following theorem holds.

**Theorem 6** Consider a function \( f \in S \) with a non-negative Fourier transform \( \hat{f} \), and such that \( f(0) = \frac{1}{2\pi} \). Then the family \( \mathcal{F}(f(tx)) \) forms a δ-family as \( t \to 0 \).

**Proof** We have:

\[
\mathcal{F}(f(tx)) = \int f(tx)e^{-i\alpha x}dx = \frac{1}{t} \int f(u)e^{-i\alpha \frac{u}{t}}du = \frac{1}{t} \hat{f}\left(\frac{\alpha}{t}\right)
\]

Moreover,

\[
\int \hat{f}(\alpha)d\alpha = 2\pi f(0) = 1.
\]

The arguments similar to those that solve the problem 4 from HW 3, complete the proof. \( \square \)

7 A fixed point of the Fourier transform

**Theorem 7** The Gauss function \( f_0(x) = e^{-\frac{x^2}{2}} \) is a fixed point of the normalized Fourier transform.

**Proof** The proof is given by a straightforward calculation in the complex domain:

\[
\hat{f}_0(\alpha) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2}-i\alpha x}dx = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x^2+2i\alpha x-\alpha^2)}e^{-\frac{\alpha^2}{2}}dx = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x+i\alpha)^2}dx := e^{-\frac{\alpha^2}{2}}C(\alpha).
\]

Now note that \( C(\alpha) \) does not depend on \( \alpha \) and equals 1! Indeed,

\[
\sqrt{2\pi}C(\alpha) = \int e^{-\frac{1}{2}(x+i\alpha)^2}dx = \int_{R+i\alpha} e^{-\frac{z^2}{2}}dz.
\]

The function \( f_0(z) \) in any strip \( \text{Im } z \in [0,\alpha] \) tends to zero as \( \text{Re } z \to \pm\infty \). By the Cauchy integral theorem, we obtain:

\[
\int_{R+i\alpha} e^{-\frac{z^2}{2}}dz = \int_{R} e^{-\frac{z^2}{2}}dz,
\]

and therefore really does not depend on \( \alpha \). Hence, \( C = C(\alpha) \)

\[
\hat{f}_0(\alpha) = Cf_0(\alpha).
\]
This implies that $f_0$ is an eigenfunction of the normalized Fourier transform. But the Fourier transform is an isometry. Hence, $C = 1$, and $\tilde{f}_0 = f$.

By the way, we obtained yet another proof of the Euler-Poisson formula:

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

Indeed,

$$C = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = 1.$$

□