These questions are not designed to be easy or difficult, but to aid our understanding during class.

1. (Sampling rates) Nyquist’s sampling rate tells us that we must sample a sound wave at, at least, two points per wavelength if we wish to exactly recover it from those samples. In this question we will see why.
   (a) Take 5 equally-spaced points on $[0, 1]$, i.e., 0, $1/4$, $1/2$, $3/4$, and 1. Show that $\cos(\pi x)$ and $\cos(7\pi x)$ evaluate to the same values at these points.
   (b) For what other values of $M$ does $\cos(M\pi x)$ evaluate to the same values as $\cos(\pi x)$ at 5 equally points.
   (c) Give an argument why part (a) and (b) shows that at least 2 points per wavelength is required.

2. (The principle of divide-and-conquer) The FFT is a divide-and-conquer algorithm. To get an idea of what this process entails, here is a problem for which divide-and-conquer is an efficient algorithm:
   Suppose you are given an array $a$ of $n$ sorted integers that has been circularly shifted $k$ positions to the right (but you do not know $k$). For example, the vector could be $a = [34, 45, 2, 12, 13, 29]$. It is simple to find the large entry of $a$ in $O(n)$ time. Describe an $O(\log n)$ algorithm. (Of course, you will only be able to read $O(\log n)$ entries of $a$.)

3. (FFT applications) The FFT is a fast algorithm to compute the matrix-vector product $Fv$ in $O(N \log N)$ operations, where $F$ is the discrete Fourier transform matrix. Show that the following operators can also be done fast, assuming you can use `fft()`: 
   \[
   [F \circ (x^T y)] v, \quad FX^T,
   \]
   where ‘$\circ$’ is the Hadamard product, $x$ and $y$ are vectors, $X$ is a matrix, and $F^T$ is the transpose of $F$.
   If you have the `ifft()` command (inverse FFT), then how would you solve
   \[
   (F + x^T y) v = b
   \]
   (Look up the Woodbury matrix formula.)