MORSE-BOTT THEORY

The goal of this talk is to introduce elements of Morse-Bott theory and some of its applications to the geometry of moment maps.

Let $f : M \to \mathbb{R}$ be a smooth function on a manifold $M$.

**Definition 1.** A point $p \in M$ is called a critical point of $f$ if the induced map $df_p : T_pM \to T_{f(p)}\mathbb{R}$ is zero. Suppose $M$ is an $n$–dimensional manifold and $(U; x_1, \cdots, x_n)$ is any coordinate chart containing $p$. If $p$ is a critical point, then $\frac{\partial f}{\partial x_i} (p) = 0$ for all $i \in \{1, \cdots, n\}$. For a critical point $p$ of $f$, the value $f(p)$ is called a critical value, otherwise $f(p)$ is called a regular value.

Let $p$ be a critical point of $f$. Define a bilinear form $H_p f$ on $T_p M$ as follows: For $v, w \in T_p M$, $H_p f (v, w) := V_p (W(f))$ where $V$ and $W$ are vector field extensions of $v$ and $w$ respectively. One can show that $H_p f$ is symmetric and well defined on $T_p M$ and is called the Hessian of $f$ at $p$. With a local coordinate system $(U; x_1, \cdots, x_n)$ containing $p$, the tangent space $T_p M$ has the basis $\frac{\partial}{\partial x_1}|_p, \cdots, \frac{\partial}{\partial x_n}|_p$ with respect to which the Hessian is represented by the matrix

$$
(0.1) \quad \left( \frac{\partial^2 f}{\partial x_i \partial x_j} (p) \right).
$$

**Definition 2.** The index of $f$ at $p$ is defined to be the index of $H_p f$, namely, the maximal dimension of a subspace of $T_p M$ on which $H_p f$ is negative definite. The nullity of $f$ at $p$ is the nullity of $H_p f$, namely, the dimension of the subspace of $T_p M$ consisting of vectors $v \in T_p M$ such that $H_p f (v, w) = 0$ for all $w \in T_p M$.

**Definition 3.** A critical point $p$ of $f$ is said to be non-degenerate if the nullity of $f$ at $p$ is trivial, that is, there is a local chart containing $p$ with respect to which the Hessian matrix given by (0.1) is non-singular. If all the critical points of $f$ are non-degenerate, then $f$ is called a Morse function.

**Lemma 4.** (Morse Lemma) Let $p$ be a non-degenerate critical point of $f$. Then there is a local coordinate system $(U; x_1, \cdots, x_n)$ containing $p$ with $x_i (p) = 0$ for all $i$ and $f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ holds throughout $U$, where $\lambda$ is the index of $f$ at $p$.

**Corollary 5.** Non-degenerate critical points are isolated. In particular, if $M$ is compact, then smooth real valued functions on $M$ have finitely many non-degenerate critical points.

1. Changing Homotopy Type

Famous motivating example: Consider a torus $M$ tangent to a plane $V$ at a point. Let $f : M \to \mathbb{R}$ be the function giving the height of $M$ above the plane $V$. Denote by $M^a$ the set of all points $x \in M$ such that $f(x) \leq a$. Note that if $a$ is a regular value of $f$, then $M^a$ is a submanifold of $M$. The height function $f$ has four critical points $p_1 < p_2 < p_3 < p_4$ which are all non-degenerate. As $a$ varies among regular values of $f$, have the following submanifolds $M^a$: 

1.
In particular, if $M$ is homeomorphic to a 2–cell.

(2) If $f(p_1) < a < f(p_2)$, then $M^a$ is homeomorphic to a 2–cell.

(3) If $f(p_2) < a < f(p_3)$, then $M^a$ is homeomorphic to a cylinder.

(4) If $f(p_3) < a < f(p_4)$, then $M^a$ is homeomorphic to a punctured torus.

(5) If $a > f(p_4)$, then $M^a$ is homeomorphic to $M$.

The non-degenerate critical points of $f$ seem to encode the information of the changing homotopy type of $M^a$. In particular, the change of homotopy type (1) $\rightarrow$ (2) is precisely the attaching of a 0–cell which is homotopic to a 2–cell. The change (2) $\rightarrow$ (3) is the attaching of a 1–cell and the resulting space is homotopic to a cylinder. The change (3) $\rightarrow$ (4) is again the attaching of a 1–cell and the resulting space is homotopic to a punctured torus. Lastly, the change (4) $\rightarrow$ (5) is the attaching of a 2–cell. Note that the dimension of the attached cell when “passing” a critical point corresponds to the index of of that critical point. These observations are generalized in the following results.

Let $f$ be a real-valued function on an $n$–dimensional manifold $M$ and let $M^a = f^{-1}(-\infty, a]$.

**Theorem 6.** If $a < b$ and $f^{-1}[a, b]$ is compact and contains no critical points of $f$, then $M^a$ is diffeomorphic to $M^b$. Furthermore, $M^a$ is a deformation retract of $M^b$, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

**Proof.** Take a Riemannian metric on $M$ and let $(\cdot, \cdot)$ be the inner product induced by the metric. Consider the vector field $\nabla f$, called the gradient vector field of $f$, that is characterized by $\langle X, \nabla f \rangle = X(f)$ for every vector field $X$. Define a smooth function $\rho : M \rightarrow \mathbb{R}$ which takes the value $1/\|\nabla f\|^2$ on $f^{-1}[a, b]$ and vanishes outside a compact neighbourhood of $f^{-1}[a, b]$. The vector field $X$ defined by $X_p := \rho(p) (\nabla f)_p$ vanishes outside a compact set and therefore generates a global flow $\varphi_t : M \rightarrow M$.

Fix $q \in M$ and consider the map $t \mapsto f(\varphi_t(q))$. If $\varphi_t(q) \in f^{-1}[a, b]$, then

$$\frac{d}{dt} (\varphi_t(q)) = \langle \frac{d\varphi_t(q)}{dt}, (\nabla f)_q \rangle = \langle X_{\varphi_t(q)}, (\nabla f)_q \rangle = 1.$$

In particular, if $a \leq f(\varphi_t(q)) \leq b$, then $f(\varphi_t(q)) = t + C$ for some constant $C$. For $q \in M$ with $f(q) = a$, then $a = f(q) = f(\varphi_0(q)) = C$ and $f(\varphi_{b-a}(q)) = b$. Hence $\varphi_{b-a}$ maps $M^a$ diffeomorphically to $M^b$ and similarly, $\varphi_{a-b}$ maps $M^b$ diffeomorphically to $M^a$. The family $r_t : M^b \rightarrow M^b$ given by

$$r_t(q) = \begin{cases} q & \text{if } f(q) \leq a \\ \varphi_{t(a-f(q))} & \text{if } a \leq f(q) \leq b \end{cases}$$

gives a deformation retract of $M^b$ to $M^a$. \hfill \Box

**Theorem 7.** Let $f : M \rightarrow \mathbb{R}$ be a smooth function, and let $p$ be a non-degenerate critical point with index $\lambda$. Setting $f(p) = c$ and suppose that $f[c-\varepsilon, c+\varepsilon]$ is compact and does not contain any critical point of $f$ other than $p$, for some $\varepsilon > 0$. Then for all sufficiently small $\varepsilon$, $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a $\lambda$–cell attached.

See [Mil] for a careful proof.
2. Morse-Bott theory

Let $f$ be a smooth real-valued function on an $n$-dimensional manifold $M$. Denote by $C_f$ the set of critical points of $f$ and let $C$ be a connected component of $C_f$. With respect to some Riemannian metric, the tangent space at every point $p \in C$ decomposes into $T_pM = T_pC \oplus N_pC$, where $N_pC$ is the normal bundle at $p$ with respect to the chosen Riemannian metric. Note that the $T_pC$ vanishes under the Hessian. Indeed, if $v, w \in T_pC$, $H_p(f)(v, w) = V_p(W(f))$ and with respect to the chosen metric, $W(f) = (W, \nabla f)$, where $\nabla f$ is the gradient vector field corresponding to $f$. Since points in $C$ are critical points, the gradient vector field vanishes on $C$. Therefore $H_p(f)$ induces a symmetric bilinear form on $N_pC$ which we denote by $h_pf$.

**Definition 8.** A smooth submanifold $C \hookrightarrow M$ is said to be a non-degenerate critical submanifold if $C \subset C_f$, $C$ is connected, and for all $p \in C$, the induced symmetric form $h_pf$ is non-degenerate. Note that $h_pf$ is non-degenerate if and only if $T_pC = \text{Ker}(H_pf)$, that is $H_pf$ be non-degenerate in the direction normal to $C$ at $p$. We say that $f$ is a Morse-Bott function if the connected components of $C_f$ are non-degenerate critical submanifolds.

**Lemma 9.** (Morse-Bott) Let $f : M \to \mathbb{R}$ be a Morse-Bott function and $C$ a connected component of $C_f$ of dimension $k$ as a manifold. Then for $p \in C$, there exists a local coordinate system $(U; \varphi = (x_1, \cdots , x_k, y_1, \cdots , y_{n-k}) : U \to V \subset \mathbb{R}^k \times \mathbb{R}^{n-k})$ containing $p$ such that $\varphi(p) = 0$, $\varphi(U \cap C) = \{(x, y) \in V : y = 0\}$ and the identity $f = f(C) - y_1^2 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_{n-k}^2$ holds throughout $U$, where $\lambda$ is the index of $h_pf$.

An immediate consequence the Morse-Bott lemma is the fact that the index of $h_pf$ is locally constant and is therefore an invariant of $C$ called the index of $C$. See [BH] for many proofs of the Morse-Bott lemma.

**Examples:**

1. Let $M = \mathbb{R}^n$ and $I, J, K$ disjoint subsets of $\{1, \cdots , n\}$ such that $I \cup J \cup K = \{1, \cdots , n\}$. Define $f : M \to \mathbb{R}$ by $f(x) = \frac{1}{2} \sum_{i \in I} x_i^2 - \frac{1}{2} \sum_{i \in J} x_i^2$. Then $f$ is Morse-Bott but not Morse.

2. The height function of a torus tangent to a plane at a point is Morse and hence Morse-Bott.

3. The height function of a torus tangent to a plane along a circle is Morse-Bott but not Morse since its critical submanifolds are circles.

3. Stable and unstable cell bundles

Now, we assume that $M$ is a compact connected manifold equipped with a Riemannian metric and $f : M \to \mathbb{R}$ is Morse-Bott. Let $\nabla f$ be the gradient vector field of $f$ on $M$ and consider its associated flow $\psi_t : M \to M$, $t \in \mathbb{R}$. Define for each critical submanifold $C_i$ the sets

\[
W_i^+ = \left\{ p \in M : \lim_{t \to +\infty} \psi_t(p) \in C_i \right\} \quad \text{and} \quad W_i^- = \left\{ p \in M : \lim_{t \to -\infty} \psi_t(p) \in C_i \right\}
\]

called the stable and unstable sets of $C_i$ respectively.

**Theorem 10.** If $f$ is Morse-Bott, then each $W_i^+$ and $W_i^-$ is a fiber bundle over $C_i$. Let $\lambda_i$ be the index of $C_i$ and $k$ its dimension. Then the fibers of the stable
and unstable sets are cells of dimension equal to \( \lambda_i \) and \( n - k - \lambda_i \) respectively. Moreover,

\[
M = \bigsqcup W^+_i \quad \text{and} \quad M = \bigsqcup W^-_i.
\]

Proof. Note that for any point \( p \in M \), if \( \lim_{t \to \pm \infty} \psi_t(p) \) exists, then it is a critical point. That such a limit always exists is a consequence of the compactness of \( M \) and the existence of “nice” neighborhoods around critical manifolds given by Morse-Bott lemma. Hence, \( M \) can be realized as a disjoint union of the stable (and unstable) sets. The maps

\[
\pi^+_i : W^+_i \to C_i \quad \text{and} \quad \pi^-_i : W^-_i \to C_i
\]

given by \( \pi^+_i(p) = \lim_{t \to +\infty} \psi_t(p) \) and \( \pi^-_i(p) = \lim_{t \to -\infty} \psi_t(p) \) respectively give the fiber bundle structure. The local description of \( f \) given by Morse-Bott lemma shows that at each \( p \in C_i \), the fibers corresponding to \( \pi^+_i \) are \( \lambda_i \)-cells and the fibers corresponding to \( \pi^-_i \) are \( (n - k - \lambda_i) \)-cells. \( \square \)

Examples:

1: Consider \( f \) as in example (1) above. Define

\[
C_K = \{ p = (p_1, \ldots, p_n) \in \mathbb{R}^n : p_k = 0 \ \text{for} \ k \in K \}
\]

\( C_1, C_2 \) similarly. Then the critical submanifold has one connected component \( C_1 = C_K \) and \( W^+_1 = C_K \sqcup C_J \) which is not \( M \). This is an example where \( M \) is not compact and the function on \( M \) is Morse-Bott but not every point converges via the gradient flow to a critical point.

2: Let \( M = \mathbb{S}^2 \) and \( f : M \to \mathbb{R} \) is the height function. Then \( f \) is a Morse function with critical submanifolds \( C_1 = \{ S \} \) and \( C_2 = \{ N \} \) consisting of the south and north pole with \( C_1 \) having index 0 and \( C_2 \) having index 2. The bundle \( W^+_1 \) over \( C_1 \) is the 0-cell \( \{ S \} \) and the bundle \( W^+_2 \) over \( C_2 \) is the 2-cell \( \mathbb{S} \setminus \{ S \} \).

Theorem 11. If \( f \) is Morse-Bott and all critical submanifolds are of even dimension and even index, then \( f \) has a unique local maxima and unique local minima.

Proof. Consider the decomposition of \( M \) into cell bundles given in Theorem 10. Then there exists some \( W^+_i \) and \( W^-_i \) of codimension zero in \( M \) hence \( W^+_i \) has \( (n - k) \)-cells as fibers over \( C_i \) and \( \lambda_i = n - k \). Similarly, \( W^-_i \) has \( (n - k) \)-cells as fibers over \( C_i \), so \( \lambda_j = 0 \).

Take the cell decomposition of \( M \) into unstable cell bundles

\[
M = W^-_1 \bigsqcup \cdots \bigsqcup W^-_s \bigsqcup W^-_{s+1} \bigsqcup \cdots \bigsqcup W^-_N
\]

where \( W^-_i \) for \( i \leq s \) correspond to index-zero critical submanifolds and for \( i > s \) correspond to critical submanifolds of index \( \geq 2 \). Let \( a_i = f(C_i) \). Then \( a_1, \ldots, a_s \) are all local minimas of \( f \). Since the codimension of \( W^-_i \) is \( \geq 2 \) for \( i > s \), \( M \bigsqcup_{i > s} W^-_i \) is connected and hence \( s = 1 \) and \( a_1 \) is the unique local minima. Conversely, one can consider the cell decomposition of \( M \) into stable cell bundles. Since there exists at least one critical submanifold of index \( n - k \), a similar argument shows that there is only one such critical submanifold. The image of \( f \) on this critical submanifold will be the unique local maxima. \( \square \)

Corollary 12. If \( f \) is Morse and the indices and dimensions of its critical submanifolds are even, then \( \pi_1(M) = 0 \).
Proof. Since $f$ is Morse, all critical submanifolds are $0$–dimensional and $M$ has a cell-decomposition with $\lambda_i$–cells for each critical point $c_i$, where $\lambda_i$ is the index of $c_i$. Theorem 6 and 7 imply that $M$ has the homotopy type of a finite CW complex (a careful proof of this is in chapter 1 of [Mil]). In particular, the zero skeleton $M^0$ has exactly one point corresponding to the unique critical point giving the relative minima. All other skeletons $M^i$ are formed by attaching cells of even dimension $\geq 2$ and hence $\pi_1(M) \cong \pi_1(M^0) = \pi_1(M^0) = 0$.

Remark. Theorem 11 is an important ingredient in understanding the geometry of images of moment maps.

References