SYMPLECTIC GEOMETRY: LECTURE 2
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1. Symplectic Manifolds

1.1. Basic definitions.

1.1. Recall: manifolds, immersion, submersion, embedding, tangent bundle, co-tangent bundle ([3]). In particular, we have partition of unity.

Let $\omega$ be a differential 2-form on $M$, i.e, an assignment of anti-symmetric bilinear 2-form $\omega_p$ on $T_pM$ for each $p \in M$, that varies smoothly in $p$. We say that $\omega$ is closed if $d\omega = 0$ where $d$ is the exterior derivative.

We say that $\omega$ is symplectic if it is closed and $\omega_p$ is symplectic (non-degenerate) for all $p \in M$.

Note: If $\omega$ is symplectic then $\dim T_pM = \dim M$ must be even.

A symplectic manifold is a pair $(M,\omega)$ where $M$ is a manifold and $\omega$ a symplectic form.

Example. Let $M = \mathbb{R}^{2n}$, or an open subset of $\mathbb{R}^{2n}$, with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic. Is $\omega_0$ closed? YES (as follows from the linearity and the multiplicative law of the exterior derivative). It is even exact: $\omega_0 = d \sum_{i=1}^{n} x_idy_i$. The ordered set

$$\left( \left( \frac{\partial}{\partial x_1} \right)_p, \ldots, \left( \frac{\partial}{\partial x_n} \right)_p, \left( \frac{\partial}{\partial y_1} \right)_p, \ldots, \left( \frac{\partial}{\partial y_n} \right)_p \right)$$

is a symplectic basis of $T_pM$.

1.2. We saw that the $n$th exterior power $\omega^n$ is a top degree non-vanishing form, i.e., a volume form. Hence $(M,\omega)$ is canonically oriented by the symplectic structure. (Does the mobius strip admit a symplectic form?)

Claim 1.1. Assume that $M$ is compact and with no boundary.
The de Rham cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ is non-zero.

Proof: otherwise $\omega^n = d\Omega$ for a $2n-1$-form. Then

$$0 = \int_{\partial M} \Omega = \int_M d\Omega = \int_M \omega^n \neq 0,$$

where the first equation is since $M$ is compact; the second by Stokes’ theorem; the third by assumption and the last inequality since $\omega^n$ is non-vanishing. (Remark: Stokes Theorem holds here since $M$ is compact.)

Therefore $[\omega]$ is non zero (i.e., $\omega$ is not exact). (Proof: $[\omega]^n = [\omega^n]$.)

In particular, if $n > 1$ there are no symplectic forms on $S^{2n}$.

There is a symplectic form (the area form) on $S^2$ and you will study it in Problem Set 2.

A symplectomorphism between $(M_1, \omega_1)$ and $(M_2, \omega_2)$ is a diffeomorphism $\phi: M_1 \to M_2$ such that $\phi^*\omega_2 = \omega_1$, i.e., . The group of symplectomorphisms of $M$ onto itself is denoted by $\text{Symp}(M, \omega)$.

**Example.** Let $\Sigma$ be an orientable 2-manifold and $\omega$ a volume form. Then $\omega$ is non-degenerate (since $\omega^n = \omega \neq 0$ everywhere) and closed (since it is a top degree form). A symplectomorphism in this case is a volume-preserving diffeomorphism. By a result of Moser, any two volume forms on a compact manifold $M$, defining the same orientation and having the same total volume are related by a diffeomorphism of $M$. In particular, every closed symplectic 2-manifold is determined up to symplectomorphism by its genus and total volume. (You will prove this result of Moser in PS3.)

1.2. **Almost complex structures.** An almost complex structure on a (real) manifold $M$ is an automorphism $J: TM \to TM$ such that $J^2 = -\text{Id}$ (i.e., it is an almost complex structure on every $T_pM$ that varies smoothly). It is integrable if it comes from a complex atlas on the manifold. An almost complex structure $J$ is compatible with a symplectic form $\omega$ if $\omega(\cdot, J\cdot)$ is a Riemannian metric on $M$, i.e., $J$ is $\omega$-compatible on every tangent space $T_pM$.

We denote by $\mathcal{J}(M, \omega)$ the space of $\omega$-compatible almost complex structures on $M$. Our proof from symplectic linear algebra can be carried fiberwise, to get a canonical surjective map $\text{Riem}(M) \to \mathcal{J}(M, \omega)$ which is a left inverse to the map $\mathcal{J}(M, \omega) \to \text{Riem}(M)$ associating to $J_p$ the corresponding inner product $\omega(\cdot, J_p\cdot)$ on $T_pM$. Since the polar decomposition is canonical, the obtained $J$ is smooth. Therefore, $\mathcal{J}(M, \omega)$ is not empty. Moreover, any $J_0, J_1 \in \mathcal{J}(M, \omega)$ can be
smoothly deformed within $\mathcal{J}(M,\omega)$: use a convex combination
\[ g_t := (1-t)g_0 + tg_1 \]
of the corresponding Riemannian metrics, and apply the polar decomposition to $(\omega, g_t)$ to obtain a smooth family of $J_t$s joining $J_0$ to $J_1$.

**Corollary 1.1.** The space $\mathcal{J}(M,\omega)$ is path-connected and not empty.

**Exercise.**
1. Fix a symplectic form $\omega$ on $M$. Let $J_0, J_1 \in \mathcal{J}(M,\omega)$ and $0 \leq t \leq 1$. Show that $J_t := (1-t)J_0 + tJ_1$ is not necessarily in $\mathcal{J}(M,\omega)$.
2. Fix an almost complex structure $J$ on $M$. Let $\omega_0$ and $\omega_1$ be symplectic forms that are compatible with $J$, and $0 \leq t \leq 1$. Show that $\omega_t = (1-t)\omega_0 + t\omega_1$ is a symplectic form that is compatible with $J$.

This exercise is in PS2.

If $J$ is integrable and $\omega$-compatible, the triple $(M,\omega,J)$ is called a Kähler manifold.

**Example.** For example, in an open set $U \subset \mathbb{C}^n (\cong \mathbb{R}^{2n})$ with coordinates $z_j = x_j + iy_j$. The almost complex structure induced from multiplication by $i$ is, in the symplectic basis:
\[ J_0 \left( \frac{\partial}{\partial x_j} \right)_p = \left( \frac{\partial}{\partial y_j} \right)_p, \]
\[ J_0 \left( \frac{\partial}{\partial y_j} \right)_p = -\left( \frac{\partial}{\partial x_j} \right)_p. \]

For a complex manifold, this defines a complex structure locally, in a chart, which is well-defined globally, since the transition maps are holomorphic (using the Cauchy-Riemann equations). [1, p.102].

**1.3. Kähler manifolds.**

Let $M$ be a complex manifold (a manifold with an atlas consisting of open subsets of $\mathbb{C}^n$ such that the transition functions are holomorphic) with a Hermitian $h$ metric on $TM$. As before, we can get a non-degenerate 2-form by assigning $\omega_p = \text{Im}(h_p)$ at every $p \in M$. If this form is closed ($d\omega = 0$) we get a Kähler structure. Do we always have a Hermitian metric on a complex manifold? Yes, using partition of unity.

**Claim 1.2.** If $(M,\omega,J)$ is a Kähler manifold, then there is a Hermitian metric $h$ on the complex manifold such that $\omega = \text{Im} h$.

Example. For example, \( \mathbb{C}^n (\cong \mathbb{R}^{2n}) \) with coordinates \( z_j = x_j + iy_j, \bar{z}_j = x_j - iy_j \). We have

\[
d\bar{z}_j \otimes dz_j = (dx_j - idy_j) \otimes (dx_j + idy_j)
\]
\[
= (dx_j \otimes dx_j + dy_j \otimes dy_j) + i(dx_j \otimes dy_j - dy_j \otimes dx_j)
\]
\[
= (dx_j \otimes dx_j + dy_j \otimes dy_j) + idx_j \wedge dy_j.
\]

Therefore the Hermitian metric

\[
h(z) = \sum_{j=1}^{n} d\bar{z}_j \otimes dz_j = g_0 + i\omega_0.
\]

(\( g_0 = \sum_{j=1}^{n}(dx_j \otimes dx_j + dy_j \otimes dy_j), \omega_0 = \sum_{i=1}^{n} dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \)). So \((\mathbb{C}^n, \omega_0)\) is Kähler.

**Claim 1.3.** Let \((M, \omega, J)\) be a Kähler manifold. Let \((N, J_N)\) be a complex manifold and \(\iota: N \to M\) a complex immersion, i.e., \(J \circ d\iota = d\iota \circ J_N\). Then \((N, \iota^*\omega, J_N)\) is a Kähler manifold.

**Proof.** It is enough to note that a complex subspace of a Hermitian vector space is Hermitian. Applying this to each \(d\iota_n(T_n N) \subset T_{i\iota(n)} M\) we see that the closed 2-form \(\iota^*\omega\ (d \circ \iota^* = \iota^* \circ d)\) is non-degenerate, and \(J_N \in \mathcal{J}(N, \iota^*\omega)\). \(\square\)

In particular, every complex submanifold of \(\mathbb{C}^n\), with the pullback of \(\omega_0\), is Kähler.

1.4. Wirtinger’s inequality (proved in a previous lecture) says that for \(X_1, \ldots, X_{2k}\) in \(\mathbb{R}^{2n}\) that are orthonormal with respect to \(g_0 = \omega_0(\cdot, J_0\cdot)\), we have

\[
|\omega_0^k(X_1 \wedge \ldots \wedge X_{2n})| \leq k!,
\]

with equality holding precisely when \(\text{span}(X_1, \ldots, X_k)\) is a complex \(k\)-dimensional subspace of \(\mathbb{C}^n\).

Therefore, if \(N\) is a smooth \(2k\)-dimensional manifold (smoothly) immersed in \(\mathbb{C}^n\), Wirtinger’s inequality implies that

\[
\int_N \frac{1}{k!} \omega_0^k \leq \int_N dV = \text{Vol}(N)
\]

with equality precisely when \(N\) is an immersed complex \(k\)-dimensional submanifold of \(\mathbb{C}^n\). (The volume is with respect to the Riemannian metric \(g_0\).)

(The proof is by partition of unity, on each chart using the Euclidean coordinate basis that we specified in the first example in the lecture.)

Note that the form \(\frac{\omega_0^k}{k!}\) on \(\mathbb{C}^n\) is exact (since \(\omega_0\) is), i.e., \(\frac{\omega_0^k}{k!} = d\alpha\).
Corollary 1.2. If $(N_1, \partial)$ and $(N_2, \partial)$ are compact $2k$-manifolds with boundary immersed in $\mathbb{C}^n$ and having the same boundary $\partial$, and $N_1$ is a complex $k$-manifold then

$$\text{Vol}(N_1) = \int_{N_1} \frac{\omega^k}{k!} = \int_{\partial} \alpha = \int_{N_2} \frac{\omega^k}{k!} \leq \text{Vol}(N_2),$$

where the equalities in the middle are due to Stokes' theorem.

Corollary 1.3. \cite[Theorem B]{4}. If a $k$-dimensional subvariety of a ball of radius $R$ in $\mathbb{C}^n$ passes through the center of the ball then its $2k$-volume is at least the volume of the unit ball in an Euclidean $2k$-space times $R^{2k}$.

This corollary is used in the proof of Gromov’s non-squeezing theorem that we will discuss later in the semester.

Example. Any almost complex structure on a (real) 2-dimensional manifold is integrable. (Theorem, proof: PS2.) As a result, an orientable 2-manifold with a volume form and a compatible complex structure is Kähler.

Example. The complex projective space

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\left(\mathbb{C} \setminus \{0\}\right) = S^{2n+1}/S^1$$

is the space of complex lines in $\mathbb{C}^{n+1}$: $\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by making the identifications $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. It is a complex manifold. One denotes by $[z_0, \ldots, z_n]$ the equivalence class of $(z_0, \ldots, z_n)$, and calls $z_0, \ldots, z_n$ the homogeneous coordinates of the point $p = [z_0, \ldots, z_n]$.

Let

$$S^{2n+1} \xrightarrow{\iota} \mathbb{C}^{n+1}$$

$$\pi \downarrow$$

$$\mathbb{CP}^n$$

where $\iota$ is an embedding and $\pi$ a projection. At every point $z \in S^{2n+1}$, we have a canonical splitting of tangent spaces

$$T_z \mathbb{C}^{n+1} = T_{\pi(z)} \mathbb{CP}^n \oplus \text{span}_\mathbb{C}\{z\}$$

as complex vector spaces. Since $T_{\pi(z)} \mathbb{CP}^n$ is a complex subspace, it is also symplectic. This induces a non-degenerate 2-form $\omega$ on $\mathbb{CP}^n$ which by construction is compatible with the complex structure. Letting $\tilde{\omega}$ be the symplectic (closed) form in $\mathbb{C}^{n+1}$, we have $\iota^* \tilde{\omega} = \pi^* \omega$. Therefore

$$\pi^* d\omega = d\pi^* \omega = d\pi^* \omega = \iota^* d\tilde{\omega} = 0,$$

showing that $\omega$ is closed. This shows that $\mathbb{CP}^n$ is a Kähler manifold. The 2-form $\omega$ is called the Fubini-Study form.
Therefore, by Claim 1.3, every non-singular (i.e., smooth) projective variety (i.e., zero locus of a collection of homogeneous polynomials) is a Kähler submanifold.

**Remark 1.1.** Every symplectic manifold admits a compatible almost complex structure but not necessarily an integrable one. (Examples: Kodaira, Thurston, McDuff and Gompf.)

### 1.3. Cotangent bundles.

Let $X$ be any $n$-dimensional manifold and $M = T^*X$ its cotangent bundle. Let

$$
\pi: M = T^*X \to X, \quad p = (x, \eta) \mapsto x, \quad \eta \in T^*_x X
$$

the bundle projection, and

$$
d\pi: TM \to TX, \quad d\pi_p: T_p M \to T_x X
$$

its tangent map. The **tautological** 1-form $\alpha$ is defined point-wise by

$$
\alpha_p(v) = \eta(d\pi_p(v))
$$

for $v \in T_p M$.

**Proposition 1.1.** The form $\alpha$ is the unique 1-form on $T^*X$ with the property that for any 1-form $\beta$ on the base $X$:

$$
\beta = \beta^* \alpha,
$$

where on the right hand side, $\beta$ is viewed as a section $\beta: X \to T^*X = M$.

**Proof.** The property holds for $\alpha$: first note that $\beta(x) = (x, \beta_x)$. So for $u \in T_x X$ we have

$$
(\beta^* \alpha)_x(u) = \alpha_{\beta(x)}(d_x \beta(u)) = \beta_x(d\pi_{(x, \beta_x)}(d_x \beta(u))) = \beta_x(u).
$$

The first equality is by definition of a pullback; the second by definition of $\alpha$; the third since $\pi \circ \beta: X \to X$ is Id, and by the chain rule.

The uniqueness is since for every $v \in T_p M \setminus \ker d_p \pi$ ($p = (x, \eta)$) there is a 1-form $\beta$ on $X$ with $\beta(x) = p$ such that $v$ is in the image of $d_x \beta$ hence, by (1.5), $\alpha$ is determined on $v$. Therefore the property determines $\alpha$ on $T_p M \setminus \ker d_p \pi$ hence, since these vectors span $T_p M$, on $T_p M$. \hfill \Box

We will use this characterization of the tautological 1-form to further understand it. In local coordinates: if the manifold $X$ is described by coordinate charts $(U, x_1, \ldots, x_n)$ with $x_i: U \to \mathbb{R}$ then at any $x \in U$, the differentials $(dx_1)_x, \ldots, (dx_n)_x$ form a basis of $T^*_x X$. Namely, if $\eta \in T^*_x X$ then $\eta = \sum_{i=1}^n \eta_i (dx_i)_x$ for real functions $\eta_1, \ldots, \eta_n$. The chart $(T^*U = \{(x, \eta) : x \in U, \eta \in T^*_x X\}, x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)$ is a coordinate chart for $T^*X$: the coordinates are the **cotangent coordinates.**
Claim 1.4. In the cotangent coordinates,
\[ \alpha = \sum_{i=1}^{n} \eta_i dx_i. \]

*Proof.* Because of Proposition 1.1, it is enough that for a 1-form \( \beta = \sum_{i=1}^{n} \beta_i dx_i \) on \( X \),
\[ \beta^* \sum_{i=1}^{n} \eta_i dx_i = \sum_{i=1}^{n} \beta^* \eta_i d\beta^* x_i = \sum_{i=1}^{n} \beta_i dx_i = \beta. \]

\[ \square \]

Theorem 1.1. Let \( M = T^*X \) and \( \alpha \) the canonical 1-form. Then \( \omega = -d\alpha \) is a symplectic form on \( M \).

*Proof.* In cotangent coordinates, \( \omega = \sum_j dx_j \wedge d\eta_j. \)

Given a diffeomorphism \( f: X_1 \to X_2 \). Then \( df: TX_1 \to TX_2 \) is a diffeomorphism and
\[ F = (df^{-1})^*: T^*X_1 \to T^*X_2 \]
is a diffeomorphism, called the cotangent lift of \( f \). For every \( \beta \in \Omega^1(X_1) \) there is a commutative diagram
\[
\begin{array}{ccc}
T^*X_1 & \xrightarrow{F} & T^*X_2 \\
\beta \uparrow & & \downarrow{(f^{-1})^*}\beta \\
X_1 & \xrightarrow{f} & X_2
\end{array}
\]

Proposition 1.2. Let \( F: T^*X_1 \to T^*X_2 \) be the cotangent lift of \( f: X_1 \to X_2 \). Then \( F \) preserves the canonical 1-form: \( F^*\alpha_2 = \alpha_1 \), hence \( F \) is a symplectomorphism: \( F^*\omega_2 = \omega_1 \).

*Proof.* It is enough to show that for every \( \beta \in \Omega^1(X_1) \) we have \( \beta^*(F^*\alpha_2) = \beta \). Indded,
\[ \beta^*(F^*\alpha_2) = (F \circ \beta)^*(\alpha_2) = (f^{-1})^* \beta \circ f)^*\alpha_2 = f^*((f^{-1})^* \beta)^*\alpha_2 = f^*(f^{-1})^*\beta = \beta, \]
where in the equality before the last we used the property that for every \( \gamma \in \Omega^1(X_2) \) we have \( \gamma^*\alpha_2 = \alpha_2 \).

\[ \square \]

This gives a natural homomorphism
\[ \text{Diff}(X) \to \text{Symp}(T^*X, \omega), \ f \to (df^{-1})^*. \]

Another subgroup of \( \text{Symp}(T^*X, \omega) \) is obtained from a homomorphism
\[ Z^1(X) \to \text{Symp}(T^*X, \omega). \]

In PS2 you will prove the following proposition.
Proposition 1.3. Let \( \sigma \in \Omega^2(X) \) be a closed 2-form on \( X \). The 2-form
\[ \omega = -d\alpha + \pi^*\sigma \]
is a symplectic form on \( T^*M \). The Liouville form of \(-d\alpha + \pi^*\sigma\) equals the Liouville form of \(-d\alpha\).

Corollary 1.4. For any manifold \( X \) with a closed 2-form \( \sigma \) there is a symplectic manifold \( (M, \omega) \) and \( \iota: X \to M \) such that \( \iota^*\omega = \sigma \).

Take \( M = T^*X \) and \( \omega = -d\alpha + \pi^*\sigma \) and \( \iota: X \to M \) the zero section embedding \( x \mapsto (x, 0) \).

1.4. Lagrangian Submanifolds and Symplectomorphisms. [1, 3.1, 3.2, 3.4].

1.5. Darboux Theorem. By the Darboux theorem, the dimension is the only local invariant of symplectic manifolds, up to symplectomorphisms. [1, Theorem 8.1, p.7]. The proof is the goal of our next lectures.

References