Martingale Difference Central Limit Theorem

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Why martingale difference CLT.

- Statisticians rely on CLTs to make inference.
- CLTs we have seen before all require independence.
- Martingale difference CLT extends the scope by taking into account the dependence.
Setup

- Martingale array:

\[ \{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n\} \]

zero-mean, square-integrable martingales for each \( n \geq 1 \).

- It can be derived from an ordinary martingale

\[ \{S_n, \mathcal{F}_n, 1 \leq n\} \]

by setting

\[
\begin{align*}
  k_n &= n, \\
  \mathcal{F}_{ni} &= \mathcal{F}_n, \\
  S_{ni} &= s_n^{-1}S_i \\
  s_n &= \left(\text{var}(S_n)\right)^{\frac{1}{2}}.
\end{align*}
\]
Martingale difference:

\[ X_{ni} = S_{ni} - S_{n,i-1}. \]

Notice

\[ E(X_{ni} | \mathcal{F}_{n,i-1}) = 0. \]

Conditional variance and squared variation of \( S_{ni} \):

\[ V_{ni}^2 = \sum_{j=1}^{i} E(X_{nj}^2 | \mathcal{F}_{n,j-1}), \quad U_{ni}^2 = \sum_{j=1}^{i} X_{nj}^2. \]
Martingale CLT

Theorem (Martingale CLT I)

Follow the notations above. Suppose $\eta^2$ is an a.s. finite r.v., and

$$\max_i |X_{ni}| \xrightarrow{p} 0, \quad \sum_i X_{ni}^2 \xrightarrow{p} \eta^2,$$

$$E\left(\max_i X_{ni}^2\right) < M < \infty,$$

$$\mathcal{F}_{ni} \subseteq \mathcal{F}_{n+1,i}.$$

Therefore

$$S_{nk_n} = \sum_i X_{ni} \xrightarrow{d} Z,$$

where $Z$ has the characteristic function

$$E(e^{itZ}) = E\left(\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right).$$
Martingale CLT

Theorem (Martingale CLT II)

Follow the notations above. Suppose

\[ E \left( \max_i |X_{ni}| \right) \to 0, \]
\[ \sum_i X_{ni}^2 \xrightarrow{P} \sigma^2, \]

then

\[ S_{nk_n} \xrightarrow{d} N(0, \sigma^2). \]
Proof of Martingale CLT

Main idea of the proof:

- Truncate the martingale array by a stopping time.
- Show that the truncation results in a negligible difference.
- Show that the truncated array converges in distribution to normal.
Truncation

- WLOG, $\sigma^2 = 1$.
- Define the stopping time
  $$T_n = \inf\{t : \sum_{i=1}^{t} X_{ni}^2 > 2\} \wedge k_n.$$
- Consider the new martingale array
  $$\{\tilde{S}_{ni} = S_{n,i \wedge T_n}, \mathcal{F}_{ni}, 1 \leq i \leq k_n\}$$
  with its differences
  $$Z_{ni} = S_{n,i \wedge T_n} - S_{n,(i-1) \wedge T_n} = X_{ni} 1\{\sum_{j=1}^{i-1} X_{nj}^2 \leq 2\}$$
  $$= X_{ni} 1\{T_n > i-1\}.$$
Recall

\[ \sum_{i=1}^{k_n} X_{ni}^2 \xrightarrow{p} \sigma^2 = 1. \]

After truncation,

\[ P(\tilde{S}_{nk_n} \neq S_{nk_n}) = P(X_{n,i} \neq Z_{n,i} \text{ for some } i \leq k_n) \]
\[ = P(T_n \leq k_n - 1) \]
\[ \leq P \left( \sum_{i=1}^{k_n} X_{ni}^2 > 2 \right) \rightarrow 0. \]

Therefore

\[ \tilde{S}_{nk_n} - S_{nk_n} \xrightarrow{p} 0, \quad \sum_{i=1}^{k_n} Z_{ni}^2 \xrightarrow{p} 1. \]
Convergence

▶ Suffices to show

\[ E \exp(it\tilde{S}_{nk_n}) \to \exp \left( -\frac{t^2}{2} \right). \]

▶ Taylor expansion

\[ \exp(ix) = (1 + ix) \exp \left( -\frac{x^2}{2} + r(x) \right), \]

where \(|r(x)| < |x|^3\). Thus

\[ \exp(it\tilde{S}_{nk_n}) = \prod_{j=1}^{k_n} \exp(itZ_{nj}) \]

\[ = \left( \prod_{j=1}^{k_n} (1 + itZ_{nj}) \right) \exp \left( -\frac{t^2}{2} \sum_{j=1}^{k_n} Z_{nj}^2 + \sum_{j=1}^{k_n} r(tZ_{nj}) \right). \]
Martingale CLT

Lemma
For $n \geq 1$, let $\{U_n\}, \{V_n\}$ be random variables satisfying the following conditions:

1. $U_n \xrightarrow{p} a$,
2. $\{V_n\}$ and $\{V_n U_n\}$ are uniformly integrable sequences,
3. $EV_n \rightarrow 1$.

Then

$$EV_n U_n \rightarrow a.$$ 

Based the lemma, let

$$V_n = \prod_{j=1}^{k_n} (1 + itZ_{nj}), U_n = \exp \left( -\frac{t^2}{2} \sum_{j=1}^{k_n} Z_{nj}^2 + \sum_{j=1}^{k_n} r(tZ_{nj}) \right).$$
Martingale CLT

Proof of the lemma:

\[ V_n(U_n - a) \text{ is u.i.} \quad \text{Suffices to show } V_n(U_n - a) \xrightarrow{p} 0. \]

\[ P(|V_n(U_n - a)| > \epsilon) \leq P(|U_n - a| > \epsilon/K) + P(|V_n| > K) \rightarrow 0. \]

\[ |V_nU_n| \leq 1, \text{ so } \{V_nU_n\} \text{ is u.i.} \]
Martingale CLT

To show

\[ U_n = \exp \left( -\frac{t^2}{2} \sum_{j=1}^{k_n} Z_{nj}^2 + \sum_{j=1}^{k_n} r(tZ_{nj}) \right) \xrightarrow{p} \exp \left( -\frac{t^2}{2} \right), \]

notice \( \sum_{i=1}^{k_n} Z_{ni}^2 \xrightarrow{p} 1 \) and

\[ \left| \sum_{j=1}^{k_n} r(tZ_{nj}) \right| \leq |t|^3 \sum_{j=1}^{k_n} |Z_{nj}|^3 \]

\[ \leq |t|^3 \sum_{j=1}^{k_n} |X_{nj}|^3 \]

\[ \leq |t|^3 \max_j |X_{nj}| \sum_{j=1}^{k_n} X_{nj}^2 \xrightarrow{p} 0. \]
Martingale CLT

To show

\[ V_n = \prod_{j=1}^{k_n} (1 + itZ_{nj}) = \prod_{j=1}^{T_n} (1 + itX_{nj}) \]

is u.i., by \(|1 + iz|^2 \leq \exp(z^2)|

\[ |V_n| = \left( \prod_{j<T_n} |1 + itX_{nj}| \right) |1 + itX_{nT_n}| \]

\[ \leq \exp \left( \frac{t^2}{2} \sum_{j<T_n} X_{nj}^2 \right) (1 + |t||X_{nT_n}|) \]

\[ \leq \exp(t^2)(1 + |t|\max_j |X_{nj}|). \]

Recall \( E(\max_j |X_{nj}|) \rightarrow 0\), so \( \max_j |X_{nj}| \) is u.i. So is \( V_n \).
Martingale CLT

To show $EV_n \to 1$, notice

$$EV_n = E \left( \prod_{j=1}^{k_n} (1 + itZ_{nj}) \right)$$

$$= E \left( E \left( \prod_{j=1}^{k_n} (1 + itZ_{nj}) \right) \bigg| F_{n,k_n-1} \right)$$

$$= E \left( E (1 + itZ_{nk_n} \big| F_{n,k_n-1}) \sum_{j=1}^{k_n-1} (1 + itZ_{nj}) \right)$$

$$= E \left( \prod_{j=1}^{k_n-1} (1 + itZ_{nj}) \right) = 1.$$
Reference

Peter Hall
*Martingale Limit Theory and Its Application.*

Sethuramas Sunder
A Martingale Central Limit Theorem.
http://math.arizona.edu/~sethuram/notes/wi_mart1.pdf