Determinizing, Forgetting, and Automata in Monoidal Categories

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Overview

monoidal categories are a natural setting to study automata

- automata based on **actions**, languages are **morphisms**

  connections to bialgebras/hopf algebras

  coassociative operations: “hidden parameters”

  extend work of Grossman & Larson,

  classical algebraic treatment of automata (e.g. Eilenberg)

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  coassociative operations: “hidden parameters”
  extend work of Grossman & Larson,
  classical algebraic treatment of automata (e.g. Eilenberg)

- equivalence proofs based on **morphisms of actions**
  completeness theorem
  determinizing = forgetting
  extend work of Kozen on Kleene algebra,
  Rutten on coalgebraic methods to prove automata equivalent

Monoidal Categories

a monoidal category $\mathcal{A}$ has:

- a bifunctor

$$(- \bullet -) \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

  which is **naturally associative** (satisfies pentagon condition)
- a **unit object** object $1 \in \mathcal{A}$

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Example

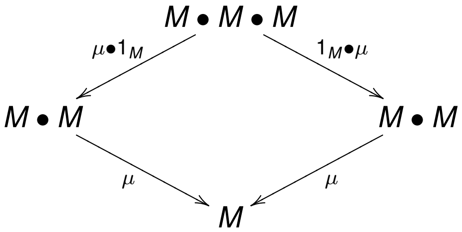
Set, $\times$, $\star$

Example

Mod_K, $\otimes$, K
(K a commutative semiring)

Monoids and Comonoids in Monoidal Categories

a monoid M has an **associative** multiplication μ and a **unit** $\eta\colon 1 \to M$



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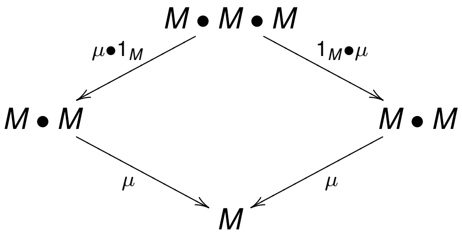
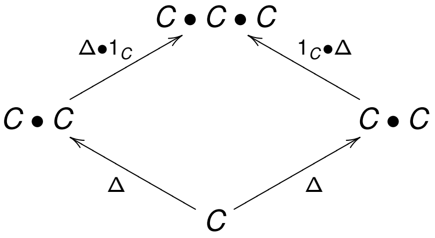


a comonoid C has a **coassociative** comultiplication Δ and a **counit** $\epsilon\colon C \to 1$



Languages as Morphisms

Fact

Let $\mathcal{A}$ be monoidal category, C a comonoid in $\mathcal{A}$, M a monoid in $\mathcal{A}$. $\mathrm{Hom}(C, M)$ *is a monoid in Set with the* ***convolution product***

$$f * g = \mu_M \circ (f \bullet g) \circ \Delta_C$$

for $f, g \in \mathrm{Hom}(C, M)$

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generalize languages to **morphisms in monoidal categories**

cf. subsets of $\Sigma^* \leftrightarrow$ formal power series

Example in Set

comonoid

C = words over Σ
$\Delta\colon C \to C \times C$
$\Delta(w) = (w, w)$

monoid

$M = \{0,1\}$
μ = multiplication in two-element Boolean algebra

for $f, g\colon C \to M$,

$$\mu \circ (f \times g) \circ \Delta(w) = f(w)g(w) \quad \text{(intersection)}$$

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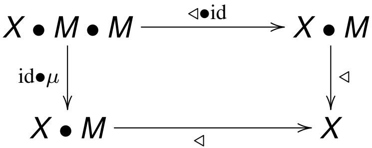
vary category, monoid, comonoid:

- union
- concatenation
- shuffle product

Actions and Automata

monoid M can **act** on object X
(right action)



Actions and Automata

monoid M can **act** on object X
(right action)

$$\begin{array}{ccc} X \bullet M \bullet M & \xrightarrow{\triangleleft \bullet \mathrm{id}} & X \bullet M \\ {\scriptstyle \mathrm{id} \bullet \mu} \downarrow & & \downarrow {\scriptstyle \triangleleft} \\ X \bullet M & \xrightarrow[\triangleleft]{} & X \end{array}$$

to turn (right) action of M on X into automaton, add:

- **start state** $\alpha \colon 1 \to X$
- **observation function** $\Omega \colon X \to O$

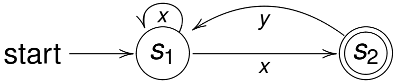
automaton **accepts** morphism

$$\rho \colon M \to O$$

$$M \xrightarrow{\cong} 1 \bullet M \xrightarrow{\alpha \bullet \mathrm{id}} X \bullet M \xrightarrow{\triangleleft} X \xrightarrow{\Omega} O$$

Example in Mod_K

“standard” nfa:



K = two-element idempotent semiring
K**-linear automaton** (automaton in Mod_K)

monoid (input)	Polynomials over $\{x, y\}$ coefficients in K
semimodule (states)	free K-semimodule on $\{s_1, s_2\}$
start	$\alpha(1_K) = 1 s_1$
observation (output in K)	$\Omega(k_1 s_1 + k_2 s_2) = k_2$
action of x	$\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Bimonoids

goal: multiplication of automata consistent with convolution

convolution product requires a **comonoid**

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bimonoid: both a monoid and a comonoid, structures compatible

$$\begin{array}{ccccc} B \bullet B & \xrightarrow{\mu} & B & \xrightarrow{\Delta} & B \bullet B \\ \downarrow{\scriptstyle \Delta \bullet \Delta} & & & & \uparrow{\scriptstyle \mu \bullet \mu} \\ B \bullet B \bullet B \bullet B & & \xrightarrow{\mathrm{id} \bullet \sigma \bullet \mathrm{id}} & & B \bullet B \bullet B \bullet B \end{array}$$

braided monoidal category: equipped with a natural isomorphism

$$\sigma : X \bullet Y \cong Y \bullet X$$

(satisfying hexagon conditions)

Multiplying Actions

Fact

The category of right actions of a bimonoid B is a monoidal category:

$$
\begin{array}{c}
X \bullet Y \bullet B \\
\big\downarrow {\scriptstyle id \bullet id \bullet \Delta} \\
X \bullet Y \bullet B \bullet B \\
\big\downarrow {\scriptstyle id \bullet \sigma \bullet id} \\
X \bullet B \bullet Y \bullet B \\
\big\downarrow {\scriptstyle \triangleleft \bullet \triangleleft'} \\
X \bullet Y
\end{array}
$$

Multiplying Automata

multiply automata: “input” a bimonoid B, “output” a monoid M

Automaton C

$\triangleleft \colon X \bullet B \to X$

$\alpha \colon 1 \to X$

$\Omega \colon X \to M$

Automaton D

$\triangleleft' \colon Y \bullet B \to Y$

$\alpha' \colon 1 \to Y$

$\Omega' \colon Y \to M$

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Automaton C	Automaton D
$\triangleleft\colon X \bullet B \to X$	$\triangleleft'\colon Y \bullet B \to Y$
$\alpha\colon 1 \to X$	$\alpha'\colon 1 \to Y$
$\Omega\colon X \to M$	$\Omega'\colon Y \to M$

product automaton $C \bullet D$

transitions	monoidal category of actions
start state	$\alpha \bullet \alpha'\colon 1 \cong 1 \bullet 1 \to X \bullet Y$
observation	$\mu_M \circ (\Omega \bullet \Omega')\colon X \bullet Y \to M$

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transitions monoidal category of actions

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Theorem (W.)

$\rho_C * \rho_D = \rho_{C \bullet D}$

(bimonoids for intersection, shuffle)

Morphisms of Automata and Proofs

morphism of actions

$f\colon X \to Y$

preserve start

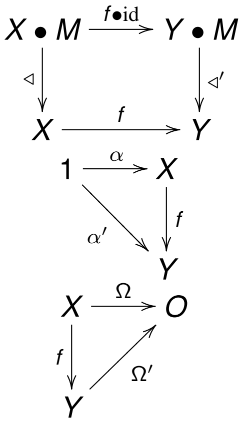
$\alpha\colon 1 \to X$

$\alpha'\colon 1 \to Y$

preserve observation

$\Omega\colon X \to O$

$\Omega'\colon Y \to O$



Fact (soundness)

Morphisms of automata preserve the morphism (language) accepted.

Completeness?

morphisms of automata = **rules of inference**
when is this system **complete**?

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consider:

	K-linear automata	**det. automata**
category	Mod_K	Set
input monoid	$K\Sigma^*$ (polynomials)	Σ^*
observation	K	$U(K)$

Theorem (W.)

There is an adjunction between the above categories. The functor from this category of K-linear automata to this category of deterministic automata is based on the forgetful functor from Mod_K to Set.

Determinizing = Forgetting

determinize K-linear automaton with state module N:

	K**-linear**	**deterministic**
start: apply adjunction between Mod_K, Set	K-linear map $K \to N$	morphism in Set $\star \to U(K)$
action of x: apply forgetful functor	K-linear map $N \to N$	morphism in Set $U(N) \to U(N)$
observation: apply forgetful functor	K-linear map $N \to K$	morphism in Set $U(N) \to U(K)$

Deterministic Automata

- yields deterministic automaton on $U(N)$ with output set $U(K)$ (n-state nfa: free semimodule on n generators)
- similar determinization for AFA
- free functor in the other direction
- can remove inaccessible states of deterministic automaton
- deterministic automata have unique minimizations

Corollary: Completeness

Let C be a K-linear automaton

$$C \xleftarrow{\epsilon} F(U(C)) \xleftarrow{F(i)} F(U(C)') \xrightarrow{F(m)} F(M(U(C)'))$$

- ϵ: counit of adjunction
- $U(C)'$: accessible part of $U(C)$
- i: inclusion $U(C)' \hookrightarrow U(C)$
- $M(U(C)')$: minimal automaton
- m: unique morphism (finality)

C and D equivalent: same minimization; combine sequences

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Theorem (W.)

For "classical" nfa's, the above proof can be constructed in PSPACE.

References

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