PROJECTS FOR PRESENTATIONS

1. The Steenrod algebra and operations (2 people)

(1) Introduce the Steenrod algebra, its construction and its properties.
(2) Use the Steenrod algebra to construct real characteristic classes.

References: [Hat02, Section 4.L], [MS74, Chapter 8]

2. Pontrjagin classes

Construct the Pontrjagin classes, discuss their relationship to other characteristic classes we have constructed, and compute the cohomology of the oriented Grassmannian.
References: [MS74, Chapter 15]

3. Even periodic theories and formal group laws

Discuss how an even periodic cohomology theory produces a formal group law. Show that these produce a theory of Chern classes. Discuss some examples.
References: See Sections 2 and 3 at `https://faculty.math.illinois.edu/~rezk/512-spr2001-notes.pdf`.

4. Clifford algebras and K-theory

Give an introduction to Clifford algebras. Discuss their connection to K-theory. Discuss Bott periodicity from the perspective of Clifford algebras.
References: [AS69b]. See also `https://www.staff.science.uu.nl/~henri105/PDF/BottPer.pdf`.

5. Vector fields on spheres

Theorem 5.1. *Let $n = k2^\nu$, for k odd. Write $\nu = 4b + c$, $0 \leq c \leq 3$ and set $\rho(n) = 8b + 2^c$. Then there exist $\rho(n) - 1$ linearly independent vector fields on spheres and no more. In other words, $TS^n \cong \epsilon^{\rho(n)-1} \oplus E$, where E has no everywhere-nonzero sections.*

References: see `http://math.harvard.edu/~ecp/latex/misc/haynes-notes/haynes-notes.pdf`. The desired result is Corollary 5.2.

6. An elementary proof of Bott periodicity

There are many different proofs of Bott periodicity. Atiyah and Bott have a particularly pretty one using "polynomial clutching functions."
References: [AB64], [Hat, Chapter 2].

7. Serre–Swan

Theorem 7.1. *Let X be compact Hausdorff, and let $C(X)$ be the ring of continuous real-valued functions on X. Then the category of vector bundles on X is equivalent to the category of finitely generated projective modules over $C(X)$.*

References: [Swa62]

8. Equivariant K-theory (2 people)

Define equivariant K-theory $K_G(X)$. Use this to prove the Thom Isomorphism Theorem and the Kunneth theorem for K-theory.

Prove that $K^*(BG) \cong R(G)$.
References: [Ati89, Section 2.7], [Seg68], [AS69a]

9. Real K-theory

Discuss Real K-theory (as opposed to real K-theory, which we discussed in class). Discuss Bott periodicity for Real K-theory. Discuss its connection to Clifford algebras. Find anything else interesting in the paper and discuss that, as well. (You might also want to read the MathSciNet review, as it is amusing.)
References: [Ati66]

References

[AB64] Michael Atiyah and Raoul Bott, *On the periodicity theorem for complex vector bundles*, Acta Math. **112** (1964), 229–247. MR 0178470

[AS69a] M. F. Atiyah and G. B. Segal, *Equivariant K-theory and completion*, J. Differential Geometry **3** (1969), 1–18. MR 0259946

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[Ati66] M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386. MR 0206940

[Ati89] ______, *K-theory*, second ed., Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989, Notes by D. W. Anderson. MR 1043170

[Hat] Allen Hatcher, *Vector bundles and K-theory*, https://www.math.cornell.edu/ hatcher/VBKT/VBpage.html.

[Hat02] ______, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 76. MR 0440554

[Seg68] Graeme Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 129–151. MR 0234452

[Swa62] Richard G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277. MR 0143225