1. Exercises

(1) Show that a vector bundle $E 	o X$ has $k$ independent sections if and only if it has a trivial $k$-dimensional subbundle.

(2) Prove that $G_n(\mathbb{R}^k)$ is homeomorphic to $G_{k-n}(\mathbb{R}^k)$.

(3) Prove explicitly that $\gamma_{12}$ is the Mobius bundle.

(4) Check that $[X,Y]$ is a group if $Y \simeq \Omega Z$ and that it is abelian if $Z \simeq \Omega Z'$. Prove that the adjunction isomorphism $[\Sigma X, Y] \cong [X, \Omega Y]$ respects this group structure.

2. Problems

(1) Let $p: TS^n \to S^n$ be the tangent bundle of $S^n$. Show that if $p$ admits an everywhere-nonzero section then the identity map is homotopic to the antipodal map $x \mapsto -x$. When $n$ is even, show that the antipodal map is homotopic to the map $(x_0, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$, which has degree $-1$. Use this to show that $TS^n$ is not trivial. (This means that $S^n$ is not parallelizable.)

(2) Let $p_1: E_1 \to B$, $p_2: E_2 \to B$ be two bundles over $B$. Prove that the bundle $E_1 \oplus E_2$ which has as it’s fiber over $b$ the vector space $p_1^{-1}(b) \oplus p_2^{-1}(b)$ is a vector bundle over $B$. Do the same for $E_1 \otimes E_2$.

Now suppose that we have a functor $\star: \text{Vect} \times \text{Vect} \to \text{Vect}$. What conditions are required on $\star$ to make it possible to define a vector bundle $E_1 \star E_2$? More generally, what if $\star$ has as its domain a subcategory of $\text{Vect}^k$? (Examples you may want to work: dual bundle, quotient bundle, orthogonal complements of subbundles.)

(3) $\mathbb{R}^n$ has a lot of good structure on it. For example, it has the Euclidean metric. Show that if $B$ is paracompact then there exists a Euclidean metric on $E$: a function $\mu: E \to \mathbb{R}$ such that its restriction to each fiber is a positive definite quadratic form.