MATH 6530 HOMEWORK 2

DUE SEPTEMBER

(1) A smooth map $f\colon M \longrightarrow N$ between smooth manifolds is a *submersion* if each Jacobian $T_x f\colon T_x M \longrightarrow T_{f(x)} N$ is surjective. Construct a vector bundle κ_f out of the kernels of the $T_x f$ and prove that it is a vector bundle. If M is Riemannian show that $TM \cong \kappa_f \oplus f^* TN$.

(2) Prove that if $n = 2^r m - 1$ and m is odd then there do not exist 2^r vector fields on $\mathbf{R}P^n$ which are everywhere linearly independent.

(3) Suppse that the n-dimensional manifold M can be immersed into $\mathbf{R}^{n+1}$. Show that for all i, $w_i(M) = w_1(M)^i$. If $\mathbf{R}P^n$ can be immersed into $\mathbf{R}^{n+1}$ show that n must be of the form $2^r - 1$ or $2^r - 2$.

(4) Let $\gamma_{1n}^{\perp}$ be the orthogonal complement to γ_{1n} in $\mathbf{R}^{n+1}$. Compute the Steifel–Whitney classes of $\gamma_{1n}^{\perp}$.

(5) The goal of this problem is to prove that cup product is Poincare dual to intersection and using it to prove that the Euler class is dual to the intersection of any section with the zero section. You may find it helpful to review Poincare duality and relative cohomology/homology. Throughout we will be working mod 2, although the entire proof works (with careful track of orientations and signs) with $\mathbb{Z}$ coefficients.

We write $[M] \in H_k(M, \partial M)$ for the fundamental class of a k-manifold M.

(a) Let B be a closed smooth oriented k-manifold, and let E be a smooth rank-n oriented vector bundle on B. Let $i\colon B \longrightarrow D(E)$ be the inclusion of B as the zero section of the disk bundle. Prove that

$$i_*[B] = [D] \frown u \in H_k(D).$$

(b) Let X be any oriented n-manifold and A any dimension-$n-i$ submanifold. Let N be a tubular neighborhood of A, which we consider as a rank-i normal bundle to A in X. Let $u \in H^i(N, N - A)$ be the Thom class of N. By excision, $H^i(N, N - A) \cong H^i(X, X - A)$; let u' be the image of u under the restriction $H^i(X, X - A) \longrightarrow H^i(X)$. If $i\colon A \longrightarrow X$ is the inclusion, prove that $i_*[A] = [X] \frown u' \in H_{n-i}(X)$ to conclude that the Poincare dual of $i_*[A]$ is exactly $u' \in H^i(X)$. Hint: consider the diagram

$$\begin{array}{ccccc}
H_n(X) \otimes H^i(X, X\backslash A) & \longrightarrow & H_n(X, X\backslash A) \otimes H^i(X, X\backslash A) & \longleftarrow & H_n(N, N\backslash A) \otimes H^i(N, N\backslash A) \\
\downarrow & & \downarrow \frown & & \downarrow \frown \\
H_n(X) \otimes H^i(X) & \xrightarrow{\ \frown\ } & H_{n-i}(X) & \longleftarrow & H_{n-i}(N)
\end{array}$$

(c) Let A and B be two submanifolds of X, of dimensions $n-i$ and $n-j$, which intersect transversely (so their intersection is a submanifold of dimension $n-i-j$). Let N_A, N_B, $N_{A\cap B}$ be tubular neighborhoods of the appropriate manifolds; note that $N_{A\cap B} \cap A$ is a tubular neighborhood of $A \cap B$ inside A. Show that the Thom class of $N_{A\cap B} \cap A$, considered as a vector bundle over $A \cap B$, is given by the pullback of the Thom class of N_B, considered as a bundle over B. (Hint: Prove that $N_{A\cap B} \cap A \cong N_B|_{A\cap B}$ canonically, as oriented vector bundles.)

(d) Let $[A]$ and $[B]$ be the classes in the homology of X given by the fundamental classes of A and B. Let $[A]^*$ and $[B]^*$ be their Poincare duals. Prove that $[A]^* \smile [B]^*$ is $[A \cap B]^*$. Thus cup product is dual to intersection.

(e) Recall that *Euler class* of $p\colon E \longrightarrow B$ is the image of the Thom class under the composition

$$H^n(E, E\backslash B) \longrightarrow H^n(E) \xleftarrow{\ p^*\ } H^n(B).$$

Prove that the Euler class of p is zero if p has an everywhere-nonzero section.

(f) Let $\psi\colon B \longrightarrow E$ be a section of p which intersects the zero section transversely, and let $Z = \psi^{-1}(0)$. Let N be a tubular neighborhood of Z in B; you may assume that, as a vector bundle, $N \cong E|_Z$. Use this to show that the Euler class of E is Poincare dual to $[Z]$ in $H^n(B)$.