1. Introducing vector bundles

Further reading: [Hat, Chapter 1], [MS74, Chapter 2].

Vector bundles (or at least, tangent bundles) appear quite naturally when one tries to work with differential manifolds, since in order to define derivatives we must define what a "tangent vector" to a manifold is. Given an n-manifold M embedded in $\mathbf{R}^N$, we can define the *tangent space* TM of M to be the set of points (x, v) with $x \in M$ and $v \in \mathbf{R}^N$ such that v is tangent to M at x. The set of points with first coordinate x is an n-dimensional vector space. Since we generally think of manifolds as existing independently of the embeddings we want an independent definition of "a family of vector spaces over a space"; this is exactly the notion of a vector bundle.

Definition 1.1 ([MS74, Chapter 2])**.** A *vector bundle* on a space B (generally called the *base space*) is a space E (generally called the *total space*) together with a map $p: E \longrightarrow B$ and the structure of a vector space on each fiber $p^{-1}(x)$ for $x \in B$, satisfying the extra condition:

(VB) There exists an integer k (the *rank* of E) such that for every point $x \in B$ there exists a neighborhood $x \in U \subseteq B$ and a homeomorphism $\varphi_x: U \times \mathbf{R}^k \longrightarrow p^{-1}(U)$. This homeomorphism must satisfy the condition that $p \circ \varphi_x = \pi_B$ (the projection onto B) and that for every $y \in U$, the restriction $\varphi_x|_{y\times\mathbf{R}^k}: \mathbf{R}^k \longrightarrow p^{-1}(y)$ is a linear homeomorphism.

We often drop p from our notation. When $n = 1$ we will call such a vector bundle a *line bundle*. For any point $x \in B$ we call $p^{-1}(x)$ the *fiber over* x.

This definition thus says that a vector bundle is a continuous family of vector spaces over B. (In some formulations, the integer k only has to exist locally, so that if B is not connected it can have different ranks over different connected components. We do not care about this in the current discussion, so we will stick to vector bundles of constant rank.)

Some important examples of vector bundles:

Example 1.2. The *trivial bundle* is the bundle $B \times \mathbf{R}^k \longrightarrow B$ where the map is just projection onto the first coordinate.

Example 1.3. As mentioned before, we can define the *tangent bundle* TM to a manifold embedded in $\mathbf{R}^n$ by taking the set of points (x, v) with $x \in M$ and v tangent to M at x. However, there is also an intrinsic definition.

For any smooth n-manifold M, TM is defined to be the set $\coprod_{x\in M} T_xM$. (Recall that T_xM, the tangent space at x, is defined as the vector space of derivations.) This comes with a natural map $p: TM \longrightarrow M$ which projects onto the first coordinte. To define the topology on TM, let $\{(U_\alpha, \varphi_\alpha: U_\alpha \longrightarrow \mathbf{R}^n)\}$ be a smooth atlas on M. The local coordinates $(x_1, \ldots, x_n)$ on U_α give local coordinates $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ on T_xM. Thus we can define a map $\widetilde{\varphi}_\alpha: p^{-1}(U_\alpha) \longrightarrow \mathbf{R}^{2n}$ by

$$(a_1x_1 + \cdots + a_nx_n, b_1\partial/\partial x_1 + \cdots + b_n\partial/\partial x_n) \longmapsto (a_1, \ldots, b_n).$$

We define the topology on TM via these maps: a subset $A \subseteq M$ is open exactly when $\widetilde{\varphi}_\alpha(A \cap U_\alpha)$ is open in $\mathbf{R}^{2n}$ for all α.

Note that this is a linear map fiberwise by definition. Also note that this construction proves not only that TM is a vector bundle over M but also that it is a $2n$-manifold.

Example 1.4. Suppose that we have a manifold M embedded in $\mathbf{R}^N$. The *normal bundle* of M is the set of points (x, v) with $x \in M$ and v orthogonal to M at x. As above, this is naturally a bundle by projecting onto the M-coordinate. This does not exist independently of the embedding.

Example 1.5. Suppose that we have a vector bundle $p: E \longrightarrow B$. For every α we have a homeomorphism $\varphi_\alpha: p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbf{R}^n$. Thus for every α, β we have a composite homeomorphism

$$(U_\alpha \cap U_\beta) \times \mathbf{R}^n \xrightarrow{(\varphi_\alpha|_{U_\alpha\cap U_\beta\times\mathbf{R}^n})^{-1}} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\beta|_{U_\alpha\cap U_\beta\times\mathbf{R}^n}} (U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

which is the identity after projection to the first coordinate and a linear homeomorphism on each fiber. Thus this map gives a smooth map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_n(\mathbf{R}).$$

This satisfies:

(1) $g_{\alpha\alpha}$ is uniformly the identity.
(2) $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$ for all $x \in U_\alpha \cap U_\beta$.
(3) $g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = 1$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Now suppose that we have a collection of such g's which satisfy these conditions. Then we can assemble a bundle E on B by taking

$$E = \coprod_\alpha U_\alpha \times \mathbf{R}^n / \sim,$$

where for any $x \in U_\alpha \cap U_\beta$ we say that $(x, v) \sim (x, g_{\alpha\beta}(x) \cdot v)$. The conditions above exactly state that $\sim$ is an equivalence relation, and the smoothness conditions on the φ_α are enforced because each $g_{\alpha\beta}$ is smooth.

For example, we can use this to construct the Mobius bundle. This is a bundle over S^1. We define it using the atlas $U_1 = S^1 \backslash \{\text{north}\}$ and $U_2 = S^1 \backslash \{\text{south}\}$. We define the function $g_{12}\colon S^1 \backslash \{\text{poles}\}$ by letting it be -1 on the part of S^1 with negative x-coordinate and 1 on the part of S^1 with positive x-coordinate.

This last example is an example of a procedure that is often done in mathematics. We take an object that we understand ($\mathbf{R}^n$, $\mathbf{C}^n$, trivial bundle, ring) glue a whole bunch of them together in a nice way, and produce a new object ((real/complex) manifold, vector bundle, scheme) which is more general and interesting, while still retaining many of the properties of the simpler object.

As always, now that we have a bunch of examples of vector bundles we want to know when two vector bundles are isomorphic.

Definition 1.6. Two vectors bundles $p_1\colon E_1 \longrightarrow B$ and $p_2\colon E_2 \longrightarrow B$ are *isomorphic* if there exists a smooth homeomorphism $f\colon E_1 \longrightarrow E_2$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\ f\ } & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & B & \end{array}$$

commutes and such that its restriction to the preimage of any point $x \in B$ is a linear isomorphism.

Note that if two vector bundles over B are isomorphic they must have the same dimension.

Remark 1.7. Note that we have not used any property of $\mathbf{R}$ when defining vector bundles. Thus we could define complex vector bundles in exactly the same way as we defined real vector bundles, but using the structure of complex vector spaces instead of real ones.

We could go further. Let F be any space. A *fiber bundle with fiber* F $p\colon E \longrightarrow B$ is a map of topological spaces such that for every point $x \in B$ there exists a neighborhood $x \in U \subseteq B$ and a map $\iota\colon U \times F \longrightarrow E$ such that $p\iota = 1_U$ and ι is a homeomorphism onto its image.

Now suppose that we want to remember some extra structure on F, such that it is a real/complex vector space, that it has an action by a group, or something else. We can write down exactly the same information, but impose the extra condition that for every $y \in U$, the map $\iota|_{y \times F}\colon y \times F \longrightarrow p^{-1}(y)$ is an isomorphism respecting this structure.

Let's look at some examples of when vector bundles are isomorphic.

Example 1.8. Let $B = S^1$ and consider TS^1. A point in TS^1 is a point in S^1 together with a vector tangent to S^1 at that point. In other words, we can write a point of TS^1 as a point $(\cos\theta, \sin\theta)$ and a vector $(-\lambda\sin\theta, \lambda\cos\theta)$. This gives us a condinuous map $TS^1 \longrightarrow \mathbf{R}^3$ given by

$$((\cos\theta, \sin\theta), (-\lambda\sin\theta, \lambda\cos\theta)) \longmapsto (\cos\theta, \sin\theta, \lambda).$$

The image of this map is exactly $S^1 \times \mathbf{R}$, so we see that TS^1 is isomorphic to the trivial bundle on S^1.

What we did here is, in effect, define a nice basis for each fiber in a way which is continuous over S^1. This let us "straighten out" the structure of TS^1 and show that it is trivial. It is actually possible to do this in general.

Definition 1.9. A *section* of a vector bundle $p: E \longrightarrow B$ is a map $s: B \longrightarrow E$ such that $p \circ s = 1_B$.

Every vector bundle has at least one section: the section which sends everything to 0. (This is called the *zero section.*)

We can use sections to prove in very simple cases that vector bundles are not isomorphic. Note that every isomorphism of vector bundles must preserve the zero section s_0, so whenever E_1 and E_2 are isomorphic vector bundles, the topological spaces $E_1 \backslash s_0(B)$ and $E_1 \backslash s_0(B)$ are homeomorphic.

Lemma 1.10. *TS^1 is not isomorphic to the Mobius bundle.*

Proof. Consider $TS^1 \backslash s_0(S^1)$ and Mobius$\backslash s_0(S^1)$. Since TS^1 is trivial, this is isomorphic to $S^1 \times (\mathbf{R}^1 \backslash \{0\})$, which is not connected. However, Mobius$\backslash s_0(S^1)$ is connected (as we all know from cutting a Mobius band down the middle). Thus these are not homeomorphic. □

We can also use sections to try and "straighten out" a bundle to show that it is trivial.

Proposition 1.11. *Let $p: E \longrightarrow B$ be an n-dimensional vector bundle. There exist n sections $s_1, \ldots, s_n: B \longrightarrow E$ such that for all $x \in B$, $s_1(x), \ldots, s_n(x)$ are linearly independent if and only if E is isomorphic to a trivial bundle.*

Proof. If E is isomorphic to $B \times \mathbf{R}^n$ then we can define $s_i: B \longrightarrow E$ by $s_i(b) = (b, e_i)$ for a fixed basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$. Then $s_1, \ldots, s_n$ are linearly independent at each point.

Conversely, suppose that the sections $s_1, \ldots, s_n$ exist. Then we define a bundle map $f: E \longrightarrow B \times \mathbf{R}^n$ in the following manner. For a point $(b, v) \in E$, write $v = \sum_{i=1}^n a_i s_i(b)$; this is always possible since $s_1(b), \ldots, s_n(b)$ are linearly independent. Then map (b, v) to $(b, (a_1, \ldots, a_n))$. This map is continuous since the s_i are, and a fiberwise isomorphism by definition. Thus it is an isomorphism of vector bundles. □

Here we implicitly used the following lemma:

Lemma 1.12 ([Hat, Lemma 1.1])**.** *A continuous map $f: E_1 \longrightarrow E_2$ between vector bundles is an isomorphism if it is a linear isomorphism on each fiber.*

Proof sketch. It suffices to check that h^{-1} is continuous. Since this is a local question it suffices to do it for trivial bundles, which we can do directly. □

We get an interesting result as a corollary. This result will end up being very important later when we consider Euler classes.

Corollary 1.13. *TS^1 has no everywhere-nonzero section.*

Proof. Suppose that TS^1 had an everywhere-nonzero section. Then by Proposition 1.11 it would be trivial. However, we just proved that it is not. □

Remark 1.14. The question of the existence of nonzero sections has a natural higher-dimensional analog: given a vector bundle $E \longrightarrow B$, how many everywhere-linearly-independent sections do there exist?

We finish up this section with an important result about homotopy groups which we will often use later:

Theorem 1.15. *Let $E \longrightarrow B$ be a fiber bundle with fiber F. Then there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_n F \longrightarrow \pi_n E \longrightarrow \pi_n B \longrightarrow \pi_{n-1} F \longrightarrow \cdots \longrightarrow \pi_1 F \longrightarrow \pi_1 E \longrightarrow \pi_n B.$$

Note that for a vector bundle, since the fibers are contractible this says that $\pi_n E \cong \pi_n B$. This makes sense since we can write down a general contraction sending each point (b, v) to (b, tv) which will be the identity at $t = 1$ and the zero section at $t = 0$.