Further reading for this section: [DL61]

We have spent some time studying Grassmannians. We know their cohomology (and thus their homology), and we have mentioned that, in general, their homotopy groups are not known. However, it turns out that these spaces satisfy a nice stability condition which allows us to calculate some of their homotopy groups. Recall that we proved earlier that $G_n \simeq BO(n)$; thus to study Grassmannians it suffices to fully understand $O(n)$.

The group $O(n)$ acts on S^{n-1}, with the stabilizer of a point isomorphic to $O(n-1)$. Since this is entirely continuous, the orbit-stabilizer theorem applies and we get that $S^{n-1} \cong O(n)/O(n-1)$. More importantly, we actually get a fiber sequence

$$O(n-1) \longrightarrow O(n) \longrightarrow S^{n-1}.$$

Thus we can look at the long exact sequence of a fibration to get an exact sequence

$$\cdots \longrightarrow \pi_{i+1}S^{n-1} \longrightarrow \pi_i O(n-1) \longrightarrow \pi_i O(n) \longrightarrow \pi_i S^{n-1} \longrightarrow \cdots$$

Since $\pi_i S^{n-1} = 0$ for $i < n-1$, we see that for $i < n-2$ we have $\pi_i O(n-1) \cong \pi_i O(n)$. Thus the low-dimensional homotopy groups do not depend on the dimension of the ambient space. We might then guess that calculating the homotopy groups of $O = \operatorname{colim} O(n)$ might be simpler than calculating the homotopy groups of $O(n)$. In actuality this ends up being significantly simpler.

Theorem 11.1 (Bott Periodicity)**.**

$$\pi_k O \cong \pi_{k+8} O \qquad \textit{and} \qquad \pi_k U \cong \pi_{k+2} U.$$

Here $U = \operatorname{colim} U(n)$.

We will be following the proof from [DL61]. There are many different proofs of this theorem, from Bott's original proof using Morse Theory to a spectral sequence argument of Moore's, to new proofs using quasifibrations of Behrens and Aguilar–Prieto. The approach that we follow has the advantage that it does not require a lot of theory, relying mostly on an understanding of algebra and some topological techniques.

The two parts of the theorem are proved in similar ways. The idea of the proof is to construct *Bott maps*

$$\begin{array}{cc} \Phi\colon BU \longrightarrow \Omega SU & \\ \Phi_1\colon BSp \longrightarrow \Omega(U/Sp) & \Phi_2\colon BO \longrightarrow \Omega(U/O) \\ \Phi_3\colon U/Sp \longrightarrow \Omega(SO/U) & \Phi_4\colon U/O \longrightarrow \Omega(Sp/U) \\ \Phi_5\colon SO/U \longrightarrow \Omega SO & \Phi_6\colon Sp/U \longrightarrow \Omega Sp. \end{array}$$

Here, Sp is the infinite symplectic group, $\operatorname{colim} Sp(n)$. Once these are constructed, the main meat of the proof is showing that they are equivalences. At that point we are done, since we get equivalences

$$U \simeq \Omega BU \longrightarrow \Omega\Omega_0 U \simeq \Omega^2 U$$

and

$$O \simeq \Omega BO \longrightarrow \Omega^2(U/O) \longrightarrow \Omega^3(Sp/U) \longrightarrow \Omega^4 Sp \simeq \Omega^5 BSp \longrightarrow \Omega^6(U/Sp) \longrightarrow \Omega^7(SO/U) \longrightarrow \Omega^8 SO \simeq \Omega^8 O.$$

Since $\pi_n \Omega^i X \cong \pi_{n+i} X$ it follows that the homomotopy groups of U and O are 2-periodic and 8-periodic, respectively, and it suffices to compute the low-dimensional homotopy groups to understand them all. (Note that this proof also proves that the homotopy groups of Sp are shifts of the homotopy groups of O.)

We will not prove real Bott periodicity. This is the harder case, and requires some more theory than the complex case, but is not much more illuminating. Thus we focus our attention on the complex case and Φ_0. This case is much simpler and more geometric, although a full discussion would still take longer than we may want to take; thus we are going to set up the proof until the key step of homology computations, and then give a broad outline of how these computations are to be done.

We now turn our attention to constructing the Bott map Φ. Let $C_{k,n} = C_k \times C_n$ be a $k+n$-dimensional complex vector space. $U(k+n)$ acts on C_{k+n}. We define a continuous familiy of linear maps, α_θ, given by

$$\alpha_{\theta kn}(z_1, z_2) = (z_1 e^{i\theta}, z_2 e^{-i\theta}) \in C_k \times C_n.$$

Note that $\alpha_{\theta kn} \in U(k+n)$, so we can think of α as a map $S^1 \longrightarrow U(k+n)$, or in other words a point in $\Omega U(k+n)$. Moreover, suppose that for $k \le k'$ and $n \le n'$ we have inclusions $C_k \hookrightarrow C_{k'}$ and $C_n \hookrightarrow C_{n'}$ which induce $j_{k'-k,n'-n}\colon C_k \times C_n \hookrightarrow C_{k'} \times C_{n'}$; then the diagram

$$\begin{tikzcd}
C_k \times C_n \arrow[r, "\alpha_\theta"] \arrow[d, "j_{k'-k,n'-n}"'] & C_k \times C_n \arrow[d, "\Omega j_{k'-k,n'-n}"] \\
C_{k'} \times C_{n'} \arrow[r, "\alpha_\theta"] & C_{k'} \times C_{n'}
\end{tikzcd}$$

commutes. Define $\widetilde{\Phi}_{k,n}\colon U(k+n) \longrightarrow \Omega U(k+n)$ by

$$T \longmapsto \left(\theta \longmapsto T\alpha_{\theta kn} T^{-1} \alpha^{-1}_{\theta kn}\right).$$

Since $\widetilde{\Phi}_{k,n}$ takes any $T \in U(k) \times U(n)$ to the trivial loop, it induces a map $\Phi_{k,n}\colon U(k+n)/U(k) \times U(n) \longrightarrow \Omega U(k+n)$.

For $k \le k'$ and $n \le n'$ the diagram

$$\begin{tikzcd}
U(k+n)/U(k) \times U(n) \arrow[r, "\Phi_{k,n}"] \arrow[d, "j_{k'-k,n'-n}"'] & \Omega SU(k+n) \arrow[d, "\Omega j_{k'-k,n'-n}"] \\
U(k'+n')/U(k') \times U(n') \arrow[r, "\Phi_{k',n'}"] & \Omega SU(k'+n')
\end{tikzcd}$$

commutes. Note that

$$U(k+n)/U(k) \cong V_n(\mathbf{R}^{k+n}),$$

where $V_n(\mathbf{R}^{k+n})$ is the Steifel manifold of n-frames in $\mathbf{R}^{k+n}$. Thus

$$U(k+n)/U(k) \times U(n) \cong V_n(\mathbf{R}^{k+n})/U(n) \cong Gr_n(\mathbf{R}^{k+n}).$$

Thus $\operatorname{colim}_k U(k+n)/U(k) \times U(n) \cong BU(n)$. As before, we know that $BU \cong \operatorname{colim}_n BU(n)$. Now, in the above diagram, set $n=1$ and $k'=k$; we then get the diagram

$$\begin{tikzcd}
\mathbf{C}P^k \arrow[r, "\Phi_{k,1}"] \arrow[d, "j_{0,n'-1}"'] & \Omega SU(k+1) \arrow[d, "\Omega j_{0,n'-1}"] \\
Gr_{n'}(\mathbf{R}^k) \arrow[r, "\Phi_{0,n'}"] & \Omega SU(k+n')
\end{tikzcd}$$

Taking $k, n' \longrightarrow \infty$ we get

$$\begin{tikzcd}
\mathbf{C}P^\infty \arrow[r, "\Phi"] \arrow[d, "J"'] & \Omega SU \arrow[d, "\Omega J"] \\
BU \arrow[r, "\Phi"] & \Omega SU
\end{tikzcd}$$

The map Φ is the Bott map.

We have the following theorem, which is a refinement of Whitehead's theorem:

Theorem 11.2 ([DL61, Theorem 1.6])**.** *Let $f\colon X \longrightarrow Y$ be a map of topological spaces which is bijective on π_0. If f is an H-map of H-spaces and $f_*\colon H_i(X;\mathbb{Z}) \longrightarrow H_i(Y,\mathbb{Z})$ is an isomorphism for all i then $f_*\colon \pi_i(X) \longrightarrow \pi_i(Y)$ is an isomorphism for all i.*

Thus in order to show that Φ induces an isomorphism on homotopy it suffices to show that it is an H-map of H-spaces and that it induces an isomorphism on homology.

Lemma 11.3. *Φ is an H-map of H-spaces.*

Proof. Note that the following diagram commutes for all k, k', n, n':

$$\begin{tikzcd}
U(k+n) \times U(k'+n') \arrow[r, "\widetilde{\Phi}_{k,n} \times \widetilde{\Phi}_{k',n'}"] \arrow[d, "\mathrm{diag}"'] & \Omega SU(k+n) \times \Omega SU(k'+n') \arrow[d, "\Omega\mathrm{diag}"] \\
U(k+k'+n+n') \arrow[r, "\widetilde{\Phi}_{k+k',n+n'}"] & \Omega SU(k+k'+n+n')
\end{tikzcd}$$

The map Ωdiag is homotopic to the loop concatenation map. The analogous diagram to the above with $\Phi_{k,n}$ instead of $\widetilde{\Phi}_{k,n}$ also commutes; if we take the colimit as k, n go to infinity we get that Φ is an H-map of H-spaces, where the H-space structure on ΩU is the loop concatentation map, and the H-space structure on BU is the block diagonal map. $\square$

Note that whenever we have a homotopy associative and homotopy unital H-space, the product map endows the homology with a ring structure. To see this, first consider any two spaces X and Y. We can define a *cross product* on the homologies:

$$H_i(X) \times H_j(Y) \xrightarrow{\times} H_{i+j}(X \times Y).$$

This is done simply by defining the product directly on the cellular chain complex and noting that the cells of $X \times Y$ are exactly products of the cells of X and the cells of Y. When X is an H-space, we get a product by composing the cross product with the multiplication map:

$$H_i(X) \times H_j(X) \xrightarrow{\times} H_{i+j}(X \times X) \xrightarrow{\mu_*} H_{i+j}(X).$$

This structure is called the *Pontrjagin ring.*

The Pontrjagin ring structure of the homology of classical Lie groups is well-known. It is particularly easy to study because there is a cell structure on these groups which allows for easy computation.

Theorem 11.4 ([Yok57, Theorem 8.1(7)])**.** *The Pontrjagin ring of $U(n)$ is given by*

$$H_*(U(n)) \cong \Lambda_{\mathbb{Z}}[e_1, e_3, \ldots, e_{2n-1}].$$

The inclusion map $U(n) \longrightarrow U(n+1)$ takes e_i to e_i.

[Yok57, Theorem 8.1] is a long list of such isomorphisms. This tells us that $H_*U \cong \Lambda_{\mathbb{Z}}[e_{2i-1} \,|\, i \geq 1]$. We are not going to prove the whole theorem, but we will show how the cell structure is constructed, as it will be useful for the rest of the section.

Sketch of cellular structure. We follow the proof in [Yok56, Section 7], as it is somewhat easier to follow.

First, note that $U(n) \cong S^1 \times SU(n)$, since we can scale the first column of any $n \times n$ unitary matrix by the inverse of the determinant. By the Kunneth theorem for homology (and by tracing through the Pontrjagin ring structure) we see that $H_*(U(n)) \cong H_*(S^1) \otimes H_*(SU(n)) \cong \Lambda_{\mathbb{Z}}[e_1] \otimes H_*(SU(n))$. Thus we focus our attention on $SU(n)$.

We think of points of $\Sigma\mathbf{C}P^{n-1}$ as pairs of points (θ, x) with $\theta \in [-\pi/2, \pi/2]$ and $x \in \mathbf{C}P^{n-1}$; we assume that x is represented as $(x_1, \ldots, x_n)$ with $|x_1|^2 + \cdots + |x_n|^2 = 1$. We define $f_n \colon \Sigma\mathbf{C}P^{n-1} \longrightarrow SU(n)$ by

$$f_n(\theta, x) = \Big(I_n - 2exp(-i\theta)\cos\theta\,(x_i\overline{x}_j)_{i,j=1}^n \Big) \begin{pmatrix} -\exp(-2i\theta) & \\ & I_{n-1} \end{pmatrix}.$$

Each of the two matrices in the product is unitary, and their product has determinant 1; when $\theta = \pm\pi/2$ it is equal to I_n, so this is a well-defined map. In addition, when $x = (1, 0, \ldots, 0)$, this is also equal to I_n. Note that if we think of $\mathbf{C}P^{k-1}$ as sitting inside $\mathbf{C}P^{n-1}$ as the first k coordinates, then we have a map $f_k \colon \Sigma\mathbf{C}P^{k-1} \longrightarrow SU(n)$ which factors through the inclusion $SU(k) \longrightarrow SU(n)$. We call the maps $f_1, \ldots, f_n$ *characteristic maps.* It is not difficult to check that f_k maps $\Sigma\mathbf{C}P^{k-1} \backslash \Sigma\mathbf{C}P^{k-2}$ homeomorphically into $SU(k) \subseteq SU(n)$.

For $n \geq k_1 > k_2 > \cdots > k_j \geq 2$ we define a map

$$f_{k_1,\ldots,k_j} \colon \Sigma\mathbf{C}P^{k_1-1} \times \cdots \times \Sigma\mathbf{C}P^{k_2-1} \longrightarrow SU(n)$$

by defining $f_{k_1,\ldots,k_j}(z_1, \ldots, z_j) = f(z_1)\cdots f(z_j)$. In the top-level this gives a cell of $SU(n)$. All of these cells together give the cellular decomposition of $SU(n)$. $\square$

From the above analysis we can see where the Pontrjagin ring structure is coming from, since cells are explicitly defined to come from products of odd-dimensional cells. In addition, from this cell decomposition we can immediately see that J will be a homology isomorphism, since as a map $U(k+1) \longrightarrow U(k+n')$ it is an isomorphism up to degree $2k$. Moreover, directly from the construction of this cell structure we get the following:

Proposition 11.5. *The map $\Sigma\mathbf{C}P^\infty \longrightarrow SU$ adjoint to Φ maps the suspension of the $2k$-dimensional cell of $\mathbf{C}P^\infty$ to the primitive cell f_{k+1}. Thus the induced map on homology takes the generator in degree $2k+1$ to the generator in degree $2k+1$.*

We have the following theorem:

Theorem 11.6 ([DL61, Theorem 2.7])**.** *Let X be an H-space such that $H_*(X)$ is a transgressively generated exterior algebra on odd generators of degree at least 3. Then $H_*(\Omega X)$ is a polynomial algebra generated by the adjoints of the generators.*

Note that this theorem applies to SU. Thus we know that $H_*(\Omega SU) \cong H_*(\Omega SU)$ is a polynomial algebra generated by the adjoint maps to the generators of $H_*(SU)$. Thus $H_*(\Omega SU)$ is a polynomial algebra on generators of even degrees. In particular, this means that the map $H_*(\mathbf{C}P^\infty) \longrightarrow H_*(\Omega SU)$ induced by the adjoint map is a *surjection on the generators of the Pontrjagin ring structure of $H_*(\Omega SU)$.*

In order to show that $\Phi_* \colon H_*(BU) \longrightarrow H_*(\Omega SU)$ is an isomorphism of Pontrjagin rings, it suffices to check $H_*(BU)$ is a polynomial ring on generators z_{2i}, and that each z_{2i} maps to the generator of degree $2i$ in the Pontrjagin ring $H_*(\Omega SU)$.

First, let us consider the H-space structure on BU. By analogy to our result about $H^*(BO(k))$, we know that $H^*BU \cong \mathbb{Z}[c_{2i} \,|\, |c_{2i}| = 2i]$. The multiplication map is given by the map which takes two matrices to their block diagonal sum. This map is exactly the map characterizing the Whitney sum of two bundles; thus on cohomology it takes the generator c_{2i} to $\sum_{j+k=i} c_{2j}c_{2k}$. This gives us the complete Hopf algebra structure on $H^*(BU)$; since the Hopf algebra structure on $H_*(BU)$ is given by the dual of this, we can conclude that $H_*(BU)$ is also a polynomial algebra on even generators z_{2i}.

We now would like to show that z_{2i} maps to the appropriate generator in $H_*(\Omega SU)$. However, note that we know that b_{2i}, the generator of $H_{2i}(\mathbf{C}P^\infty)$ maps to this generator in $H_*(\Omega SU)$; thus it suffices to check that the image of b_{2i} under J is z_{2i}. Note that J is the inclusion $\mathbf{C}P^\infty \cong BU(1) \longrightarrow BU$. By the general theory of Chern classes, $J^*(c_{2i}) = c_2(\mathbf{C}P^1) = b_2 \neq 0$ if $i = 1$ and

0 otherwise. Thus $J_*(b_{2i}) = z_{2i}$ (since otherwise pushing forward, dualizing and then pulling back would give the wrong result), and the proof is complete.

We now note that $U \cong S^1 \times SU$. Since Ω is a right adjoint, we have

$$\Omega U \cong \Omega(S^1 \times SU) \cong \Omega S^1 \times \Omega SU \cong \mathbb{Z} \times \Omega SU.$$

We can therefore restate the Bott map as an equivalence

$$\mathbb{Z} \times BU \xrightarrow{\sim} \Omega U$$

by simply mapping it appropriately on components.

The real case is similar, with similar definitions for the Bott maps. However, the homology computations become somewhat more complicated, and the proof must be done in two stages: first showing that the maps analogous to ϕ_* and ι_* are isomorphisms in mod p homology for all primes p, and then lifting this to imply that they are isomorphisms in homology with $\mathbb{Z}$ coefficients. The case when $p = 2$ ends up being more complicated than the general case. For the interested reader these are explained in detail in [DL61].

We can use this to compute the homotopy groups of U. By the theorem, if we can compute π_0 and π_1 we would know all of the homotopy groups. Analogously to the observation about $O(n)$ at the beginning of this section, $\pi_i U(n-1) \cong \pi_i U(n)$ for $i < 2n-2$. Thus to find all of the homotopy groups up to π_1 it suffices to consider $n = 2$. But $U(2) \cong S^1 \times SU(2) \cong S^1 \times S^3$, which has $\pi_0 = 0$ and $\pi_1 \cong \mathbb{Z}$. Thus

$$\pi_{\text{even}} U \cong 0 \qquad \text{and} \qquad \pi_{\text{odd}} U \cong \mathbb{Z}.$$

Alternately, we can look at BU and note that the result says that

$$BU \simeq \Omega U \simeq \Omega^2 BU.$$