14. Adams operations and the Splitting Principle

Further reading: [HatB, Chapter 2]

Theorem 13.7 relies very heavily on an important general principle:

Theorem 14.1 (Splitting Principle)**.** *For any bundle $p\colon E \longrightarrow X$ with X compact Hausdorff, there exists a compact Hausdorff X' with a map $f\colon X' \longrightarrow X$ such that f^*E splits as a sum of line bundles, and the induced map $f^*\colon K^0(X) \longrightarrow K^0(X')$ is injective.*

That this theorem means is that if we want to make constructions and prove relations for general bundles, it is often enough to show them for sums of line bundles.

Assuming the Splitting Principle we can now prove Theorem 13.7.

Proof of Theorem 13.7. Note that the pullback of a Whitney sum of vector bundles is another Whitney sum of vector bundles. Property (1) holds because it holds for the λ^k.

Suppose we are given two vector bundles E and E', and we wish to compute $\psi^k(E \oplus E')$. Let $f\colon X' \longrightarrow X$ be a map that splits E; then the pullback of $E \oplus E'$ is $L_1 \oplus \cdots \oplus L_n \oplus f^*E'$. Then let $f'\colon X'' \longrightarrow X'$ be a map that splits f^*E'; this pulls back to $L_1 \oplus \cdots \oplus L_n \oplus L_1' \oplus \cdots \oplus L_{n'}'$. By construction we know that

$$\psi^k(L_1 \oplus \cdots \oplus L_{n'}') = \psi^k(L_1) + \cdots + \psi^k(L_{n'}')$$

holds; thus $\psi^k(E \oplus E') = \psi^k(E) + \psi^k(E')$, since we have the following commutative diagram:

$$\begin{array}{ccc} K^0(X) & \xrightarrow{\psi^k} & K^0(X) \\ {\scriptstyle (f'f)^*}\big\downarrow & & \big\downarrow{\scriptstyle (f'f)^*} \\ K^0(X'') & \xrightarrow{\psi^k} & K^0(X'') \end{array}$$

Thus on bundles ψ^k is additive. Since $K^0(X)$ is a group completion and ψ^k is additive on generators, it is also additive on all of $K^0(X)$.

Thus we now know that ψ^k is an additive homomorphism. Property (2) holds by definition. Then $E \otimes E'$ pulls back along $f'f$ to $\bigoplus L_i \otimes L_j'$, which is a sum of line bundles; thus

$$\psi^k(\bigoplus L_i \otimes L_j') = \bigoplus L_i^k \otimes {L_j'}^k = \Big(\bigoplus L_i^k\Big) \otimes \Big(\bigoplus {L_j'}^k\Big) = \psi^k(E)\psi^k(E').$$

Thus ψ^k is a ring homomorphism.

Property (3) follows similarly from the Splitting Principle.

For Property (4), note that $\psi^p(E)$ pulls back to

$$\sum L_i^p \equiv \Big(\sum L_i\Big)^p \pmod{p}.$$

It remains to check Property (5). First suppose that $n = 1$. We know that $\widetilde{K}^0(S^2)$ is generated by $H - 1$. We have

$$\psi^k(H-1) = H^k - 1 = (1 + (H-1))^k - 1 = 1 + k(H-1) - 1 = k(H-1),$$

since $(H-1)^i = 0$ for $i > 1$. Property (5) then follows from the fact that $\widetilde{K}^0(S^{2n}) \cong \widetilde{K}^0(S^2) \otimes \cdots \otimes K^0(S^2)$, with the Adams operations acting termwise.[k] ☐

[k]We did not actually prove this last fact. It *should* follow from our proof of Bott periodicity, as it follows from the standard proof of Bott periodicity. It follows from the proof of Bott periodicity which will be one of the final projects.

Thus we have now reduced the problem of proving Adams operations to that of proving the splitting principle. The idea behind the splitting principle lies again in familiar map $BU(n) \longrightarrow BU(n+1)$. We showed that if we pull back the canonical $n+1$-plane bundle along this map we get the canonical n-bundle plus a trivial bundle. The idea of the Splitting Principle is to generalize this idea to any bundle.

Somewhat more concretely, let $p\colon E \longrightarrow X$ be a rank-n vector bundle. Let $F(E)$ be the space whose points are n-tuples of orthogonal lines in a fiber of p. Then there is a natural projection $g\colon F(E) \longrightarrow X$. When we form g^*E each fiber will have a canonical choice of splitting into lines, and thus the bundle will split into line bundles.

To do this more rigorously, we need two ingredients:

Proposition 14.2. *As a ring,* $K(\mathbf{C}P^n) \cong \mathbb{Z}[L]/(L-1)^{n+1}$, *where* L *is the canonical line bundle over* $\mathbf{C}P^n$.

Theorem 14.3 (Leray–Hirsch)**.** *Let* $p\colon E \longrightarrow B$ *be a fiber bundle with* E *and* B *compact Hausdorff and with fiber* F *such that* K^*F *is free. Suppose that there exist classes* $c_1, \ldots, c_k \in K^*(E)$ *which restrict to a basis in each fiber* F *of* p. *If* F *is a finite cell complex with cells of only even dimensions*[1] *then as a module,* $K^*(E)$ *is a free module over* $K^*(B)$ *with basis* $\{c_1, \ldots, c_k\}$.

Using these, we can now prove the Splitting Principle:

Proof of Splitting Principle. We prove this by induction on the rank of E. Clearly, when the rank is 0 or 1 we are done. Now suppose that the theorem holds for any bundle of rank at most $n-1$.

Let $P(E)$ be the projectivization of p, so that the fibers of $g\colon P(E) \longrightarrow X$ are the spaces $\mathbf{C}P^n$. We can think of $P(E)$ as the space of lines in E, where we consider a one-dimensional subspace a "line" if it lies in a single fiber. Note that $P(E)$ is compact. There is a canonical line bundle $L \longrightarrow P(E)$ which takes over each point the line that it represents. Consider $[1], [L], [L^2], \ldots, [L^n] \in K^*(P(E))$. Over each fiber these restrict to the powers of the canonical line bundle in $\mathbf{C}P^n$; by Proposition 14.2 this is a basis for $K^*(\mathbf{C}P^n)$. Thus the Leray–Hirsch Theorem applies, and $K^*(P(E))$ is a free $K^*(X)$-module with basis $1, [L], \ldots, [L^n]$. In particular, this means that the induced map $p^*\colon K^*(X) \longrightarrow K^*(P(E))$ is an injection, since 1 is a basis vector.

Now consider the pullback bundle $g'\colon g^*E \longrightarrow P(E)$. This contains L as a subbundle, so it splits as $L \oplus E'$ with the rank of E' equaling $n-1$. Thus there exists a space X' such that pulling back E' along $X' \longrightarrow P(E)$ is an injection on K^* and the pullback bundle splits into line bundles. But then the pullback of $L \oplus E'$ must also split into line bundles, and we are done. ☐

Note that this proof actually tells us how to construct the space X': it is the space $F(E)$ whose points are n-tuples of orthogonal lines in E. This space is called the *flag bundle*.

We omit the proof of the Leray–Hirsch Theorem and the computation of the K-theory of complex projective space. We will comment that the Leray–Hirsch Theorem holds in other cohomology theories as well, and was originally proved for ordinary cohomology. In that form it is a mild generalization of the Thom isomorphism. For a proof of this, see for example [KT06, Theorem 3.1].

[1] This can also be replaced by a condition on B.