5. Characteristic classes

Recall that the reason we went on the whole tangent to cohomology is to find computable invariants of vector bundles. We are now prepared to construct these:

Definition 5.1. A *characteristic class* for n-dimensional real vector bundles is a function assigning to each vector bundle $p\colon E \longrightarrow B$ a class $\xi(E) \in H^i(B;\mathbb{Z}/2)$ which satisfies the following properties:

(I) $\xi(E)$ depends only on the isomorphism class of p.

(P) For any map $f\colon B' \longrightarrow B$, $\xi(f^*(E)) = f^*(\xi(E))$.¶

Lemma 5.2. *Characteristic classes for n-dimensional bundles correspond to elements of $H^*(G_n;\mathbb{Z}/2)$.*

Proof. Since ξ assigns each rank n bundle an element in the cohomology of the base, it must do the same for the universal bundle. Suppose $\xi(\gamma_n) = \alpha \in H^i(G_n;\mathbb{Z}/2)$. Then for any other bundle $p\colon E \longrightarrow B$ we know that there exists a map $f\colon B \longrightarrow G_n$ such that p is the pullback $f^*(\gamma_n) \longrightarrow B$. Then by property (2) we must have $\xi(p) = f^*(\xi(\gamma_n))$, so ξ is uniquely determined by $\xi(\gamma_n)$.

On the other hand, suppose $\alpha \in H^i(G_n;\mathbb{Z}/2)$. For any bundle $p\colon E \longrightarrow B$ we define $\xi(p) = f^*(\alpha)$, where $f\colon B \longrightarrow G_n$ is a map characterizing p. This clearly satisfies (1), since any two isomorphic bundles have homotopic classifying maps, and homotopic classifying maps induce the same map on cohomology. This satisfies (2) by definition, since if $f\colon B \longrightarrow G_n$ characterizes $p\colon E \longrightarrow B$ and $g\colon B' \longrightarrow B$ is any map then fg characterizes $g^*(E)$. □

We know that the cohomology of G_n is a polynomial ring on the variables w_i with $|w_i| = i$. In addition, we know that once w_i appears in $H^*(G_n)$ it is uniquely determined for $H^*(G_{n+k})$ for all $k \geq 0$. Thus a characteristic class for bundles of dimension n determines a characteristic class for all bundles of dimension *at least* n. Given that we know good generators for $H^*(G_n)$ and these come with nice well-defined gradings, it makes sense to name them.

Definition 5.3. The i-th *Stiefel–Whitney class* is the characteristic class associated to $w_i \in H^*(G_n)$ (where when $n < i$ we set $w_i = 0$).

Note that this makes sense for *all* vector bundles. Let us explore some consequences of this definition.

Lemma 5.4. *For any vector bundle $p\colon E \longrightarrow B$, $w_i(E \oplus \epsilon^k) = w_i(E)$ for all i.*

Proof. It suffices to check this for $k = 1$. Suppose that E is a rank n vector bundle, and let $f\colon B \longrightarrow G_n$ be its classifying map. Then we have a pullback square

$$\begin{tikzcd}
E \oplus \epsilon^1 \arrow[r] \arrow[d] & \gamma_n \oplus \epsilon^1 \arrow[d] \\
B \arrow[r, "f"] & G_n
\end{tikzcd}$$

Thus $w_i(E \oplus \epsilon^1) = f^*(w_i(\gamma_n \oplus \epsilon^1))$. In particular, it suffices to check that $w_i(\gamma_1 \oplus \epsilon^1) = w_i(\gamma_n)$.

The classifying map of the $n+1$-bundle $\gamma_n \oplus \epsilon^1$ on G_n corresponds to the class of the projection $S(\gamma_{n+1}) \longrightarrow G_{n+1}$ in $[G_n, G_{n+1}]$. To prove the second part it suffices to check that the pullback of γ_{n+1} along the projection $\pi\colon S(\gamma_{n+1}) \longrightarrow G_{n+1}$ is isomorphic to the pullback of $\gamma_n \oplus \epsilon^1$ along

$p\colon S(\gamma_{n+1}) \xrightarrow{\sim} G_n$. The points of $\pi^*\gamma_{n+1}$ are triples (ω, v, w) with $v, w \in \omega$ and v a unit vector. This has a section sending (ω, v) to (ω, v, tv); thus this is isomorphic to the sum of the bundles (ω, v, tv) and (ω, v, w), with $w \in v^\perp \cap \omega$. We get the pullback square

¶Note that the two f^* denote different things: in the first case it denotes the pullback bundle and in the second the map induced on cohomology.

$$
\begin{tikzcd}
(\omega, v, w) \arrow[r] \arrow[d] & (\omega \cap v^{\perp}, w) \arrow[d] & \gamma_n \arrow[l, "\cong"'] \\
S(\gamma_{n+1}) \arrow[r, "\sim", "(\omega, v) \mapsto \omega \cap v^{\perp}"'] & G_n &
\end{tikzcd}
$$

where the map along the bottom is p. Adding a trivial bundle to the top row of the diagram gives the desired result. ☐

Apart from being nicely defined for all vector bundles, the Stiefel–Whitney classes satisfy an extra beautiful relation called the Whitney sum formula.

Theorem 5.5 (Whitney sum formula)**.** *For vector bundles E and E' over a common base B, let $E \oplus E'$ denote the bundle which is the fiberwise direct sum of E and E'. Then*

$$w_i(E \oplus E') = \sum_{j+k=i} w_j(E) \smile w_k(E').$$

Proof. Consider the bundle $\gamma_n \times \gamma_n$ on $G_m \times G_n$. We begin by computing $w_i(\gamma_n \times \gamma_m)$. The classifying map of $\gamma_n \times \gamma_m$ is the map $\oplus\colon G_n \times G_m \longrightarrow G_{n+m}$ from Example 3.9. By the Kunneth formula [Hat02, Theorem 3.16],

$$H^*(G_n \times G_m) \cong \mathbb{Z}/2[w_{n1}, \dots, w_{nn}] \otimes \mathbb{Z}/2[w_{m1}, \dots, w_{mm}].$$

Here, w_{nj} is the pullback of w_j along $G_n \times G_m \longrightarrow G_n$, and w_{mk} is the pullback of w_k along $G_n \times G_m \longrightarrow G_m$. Writing $H^*(G_{n+m}) = \mathbb{Z}/2[w_1, \dots, w_{n+m}]$ we see that $\oplus^* w_i = q_i(w_{n1}, \dots, w_{mm})$ for some unique polynomial q_i.

We'll prove by induction on $m + n$ that

$$\oplus^* w_i = \sum_{j+k=i} w_{nj} w_{mk}.$$

Let $g\colon G_{n-1} \longrightarrow G_n$ be a map that induces $\eta\colon H^*(G_n) \longrightarrow H^*(G_{n-1})$; thus this map classifies the bundle $\gamma_{n-1} \oplus \epsilon^1$ on G_{n-1}. Then

$$(g \times 1)^* q_i(w_{n1}, \dots, w_{mm}) = w_i(\gamma_{n-1} \oplus \epsilon^1 \oplus \gamma_m) = w_i(\gamma_{n-1} \oplus \gamma_m),$$

by Lemma 5.4. The right-hand bundle is now over $G_{n-1} \times G_m$. By induction, the right-hand side is equal to $\sum_{j+k=i} w_{(n-1)j} w_{mk}$. We also know that g^* takes w_{nj} to $w_{(n-1)j}$ when $j < n$ and to 0 otherwise. Thus

$$\oplus^* w_i \equiv \sum_{j+k=i,\ j<n} w_{nj} w_{mk} \pmod{w_{nn}}.$$

Symmetrically, we can also conclude that

$$\oplus^* w_i \equiv \sum_{j+k=i,\ k<m} w_{nj} w_{mk} \pmod{w_{mm}}.$$

Thus by the Chinese Remainder Theorem,

$$\oplus^* w_i \equiv \sum_{j+k=i} w_{nj} w_{mk} \pmod{w_{nn} w_{mm}}.$$

When $i < m + n$ we know that this must be equality, since $w_{nn} w_{mm}$ has grading $m + n$. When $i > m + n$ $w_i = 0$ and so its pullback must also be 0; this also follows from this formula. Therefore it simply remains to check that $\oplus^* w_{m+n} = w_{mm} w_{nn}$.

It turns out that in general the Thom class of $E \times E'$ over $B \times B'$ is the product of the two Thom classes. (This can be seen directly from the Kunneth isomorphism and the property of the

Thom class.) More concretely, the Thom class in $\widetilde{H}^*(D(E \times E')/S(E \times E')) \cong \widetilde{H}^*(D(E)/S(E)) \otimes \widetilde{H}^*(D(E')/S(E'))$ is the product of the Thom class in $H^*(D(E)/S(E))$ and the Thom class in $H^*(D(E')/S(E'))$. Since the $\mathbb{Z}/2$-Euler class is obtained as a pullback of the Thom class, it follows that the $\mathbb{Z}/2$-Euler class of $E \times E'$ is the product of the $\mathbb{Z}/2$-Euler class of E and the $\mathbb{Z}/2$-Euler class of E'.

The element w_{mm} is exactly the $\mathbb{Z}/2$-Euler class of γ_m, and the element w_{nn} is exactly the $\mathbb{Z}/2$-Euler class of γ_n; thus the Euler class of $\gamma_m \times \gamma_n$ is $w_{mm}w_{nn}$. On the other hand, $\oplus^* w_{m+n}$ is the pullback of the $\mathbb{Z}/2$-Euler class of γ_{m+n} along the classifying map, so it is the $\mathbb{Z}/2$-Euler class of $\oplus^* w_{m+n}$, and the desired result follows.

Now let E and E' be two vector bundles over B, and consider the bundle $E \oplus E'$. If E has classifying map f and E' has classifying map f', then the classifying map of $E \oplus E'$ is

$$B \xrightarrow{\Delta} B \times B \xrightarrow{f \times f'} G_m \times G_n \xrightarrow{\oplus} G_{m+n}.$$

Then

$$\begin{aligned} w_i(E \oplus E') &= (\oplus(f \times f)\Delta)^* w_{m+n} = \Delta^*(f \times f')^* \oplus^* w_{m+n} \\ &= \Delta^*(f \times f')^* \sum_{j+k=i} \underbrace{w_{mj}w_{nk}}_{\in H^*(G_m \times G_n)} = \Delta^* \sum_{j+k=i} \underbrace{w_j(E)w_k(E')}_{\in H^*(B \times B)} \\ &= \sum_{j+k} w_j(E)w_k(E') \in H^*(B). \end{aligned}$$

□

Remark 5.6. We could define the *total Whitney class* of a vector bundle to be $w = 1 + w_1 + w_2 + \cdots$; note that this is well-defined because for every bundle only finitely many of these are nonzero. Then the Whitney sum formula says that $w(E \oplus E') = w(E) \smile w(E')$. However, this is a *terrible* statement to make, since it involves adding up things in different cohomology classes (which really shouldn't happen). It's possible to use equivariant cohomology to fix it, but for now we're just going to look at these classes separately so that we don't need to worry about it.

To sum up our discussion so far, we have shown that the Steifel–Whitney classes satisfy the following:

1. For every $j \geq 0$ you can assign a Steifel–Whitney class $w_j(E) \in H^j(B)$. (We define $w_0(E) = 1$.) $w_i(E) = 0$ if i is greater than the rank of E.
2. Given any map $f \colon B' \longrightarrow B$ and any bundle $E \longrightarrow B$, $w_j(f^*E) = f^* w_j(E)$.
3. For any bundles E and E' over B, we have
$$w_i(E \oplus E') = \sum_{j+k=i} w_j(E)w_k(E').$$
4. For the universal bundle $\gamma_n \longrightarrow G_n$, $w_n(\gamma_n) \neq 0$.

It turns out that these four properties uniquely characterize the Steifel–Whitney classes. (In fact, we only need to assume the fourth property for $n = 1$.) Milnor and Stasheff's approach to characteristic classes is to define these axiomatically and use them to do some interesting computations; they only prove that they exist later. (In addition, their construction uses Steenrod squares which, while very interesting, are beyond the scope of this course.)

Remark 5.7. As before, this works exactly the same way for complex vector bundles, except that all of the above classes live in $\mathbb{Z}$-coefficient cohomology. They are called the *Chern classes*.