6. Computations with characteristic classes

So now we have proved that Steifel–Whitney classes exist! We will now do some calculations wtih them. In fact, we will only use axioms (1)-(4) that we proved in the previous section. This formulation is very convenient, since it means that we don't need to worry about exactly how we constructed these classes. We follow [MS74, Section 4].

Proposition 6.1. *If $E \cong E'$ then $w_i(E) = w_i(E')$ for all i.*

Proposition 6.2. *For all $i > 0$ and any base B, $w_i(\epsilon^k) = 0$.*

Proof. Since ϵ^k is trivial it is the pullback of the trivial k-bundle over a point. The cohomology of a point is trivial above degree 0. □

Proposition 6.3. *If E is a rank-n bundle over B with an everywhere-nonzero section s then $w_n(E) = 0$. If E has k everywhere-independent sections then*

$$w_n(E) = w_{n-1}(E) = \cdots = w_{n-k+1}(E) = 0.$$

Proof. Given k everywhere-independent sections $s_1, \ldots, s_k$, E contains a trivial subbundle E' given by the spans of $s_1, \ldots, s_k$. Letting $E'' = (E')^\perp$, we see that $E \cong E' \oplus E''$. Then $w_i(E) = w_i(E'')$ by the Whitney sum formula and the fact that $w_i(E') = 0$ for $i > 0$. Since E'' has rank $n - k$, all Steifel–Whitney classes above dimension $n - k$ must be 0. □

Thus we see that we can use Steifel–Whitney classes to put a bound on the number of everywhere-independent sections that a vector bundle can have.

Proposition 6.4. *For every k there exists a polynomial q_k such that whenever $E \oplus E' \cong \epsilon^n$, $w_k(E') = q_k(w_1(E), \ldots, w_i(E))$.*

Proof. We construct q_k inductively. When $k = 0$ we have $w_0(E) = w_0(E') = 1$. For $k > 0$ we have $w_k(\epsilon^n) = 0$. Thus for $k = 1$ we have $w_1(E) + w_1(E') = 0$, by the Whitney sum formula. Thus $q_1 = -w_1(E)$. Now suppose that polynomials $q_1, \ldots, q_{k-1}$ exist. By the Whitney sum formula we have

$$w_k(E) + w_{k-1}(E)w_1(E') + \cdots + w_1(E)w_{k-1}(E') + w_k(E') = 0.$$

Solving for $w_k(E')$ and plugging in we have

$$w_k(E') = -\sum_{i=1}^{k} w_i(E) q_{k-i}(w_1(E), \ldots, w_{k-i}(E)),$$

which gives us the polynomial q_k. □

Definition 6.5. We write $\overline{w}_i(E)$ for $q_i(w_1(E), \ldots, w_i(E))$.

As a special case, we get the following:

Lemma 6.6 (Whitney duality theorem)**.** *Let TM be the tangent bundle of a manifold M in Euclidean space, and let ν be the normal bundle. Then*

$$w_i(\nu) = \overline{w}_i(TM).$$

In particular, note that the characteristic classes of the normal bundle are independent of the choice of embedding.

The fact that it is possible to find a bundle that adds to a trivial bundle is not restricted to tangent bundles of manifolds.

Proposition 6.7. *For any bundle $p\colon E \longrightarrow B$ where B is compact there exists a bundle E' such that $E \oplus E' \cong \epsilon^k$ for some k.*

Thus dual Steifel–Whitney classes make sense for all bundles over compact bases.

Proof. As we discussed in the proof of Theorem 3.3, it will suffice to construct a map $g\colon E \longrightarrow \mathbf{R}^N$ which is a linear injection on fibers. Then this will produce an embedding of E into $B \times \mathbf{R}^n$, and the orthogonal complement of each fiber of E will give the desired bundle.

We proceed as in the proof of Theorem 3.3, except that we must have a finite open cover so that we can embed into a finite-dimensional space. For each point $x \in B$ there exists a U_x over which E is trivial. By Urysolhn's Lemma there is a map $\varphi_x\colon B \longrightarrow [0,1]$ which is 0 outside U_x and nonzero at x. Then $\{\varphi_x^{-1}(0,1]\}$ is an open cover of B; since B is compact, it has a finite subcover $\{\varphi_{x_i}^{-1}(0,1]\}_{i=1}^k$. Write $h_i\colon p^{-1}(U_{x_i}) \longrightarrow U_{x_i} \times \mathbf{R}^n \xrightarrow{pr_2} \mathbf{R}^n$ for the composition of the local trivialization at x_i and the projection onto the second coordinate. When we multiply by $\varphi_i(p(e))$ we can extend this to all of E by 0. We then define

$$g(e) = (\varphi_1(p(e))h_1(e), \ldots, \varphi_k(p(e))h_k(e)) \in \mathbf{R}^{nk}.$$

□

Remark 6.8. What this proposition says is that, as far as Steifel–Whitney classes are concerned, bundles have "inverses" with respect to $\oplus$. We will see this perspective again when we discuss K-theory.

Example 6.9. Consider TS^n. Note that when S^n is embedded in $\mathbf{R}^{n+1}$, the normal bundle is trivial. Thus $w_i(\nu) = 0$ for $i > 0$. Since the sum of the two bundles is also trivial, we must have $w_i(TS^n) = 0$ for $i > 0$. However, we showed on the homework that TS^n is nontrivial. (In fact, if we were working with $\mathbb{Z}$ coefficients we'd see that $w_n(TS^n) = 2[S^n]$.)

Example 6.10. Consider the universal line bundle γ_{1n} over $\mathbf{R}P^n$. We have $H^*(\mathbf{R}P^n) \cong \mathbb{Z}/2[x]/(x^{n+1})$. Note that $w_i(\gamma_{1n}) = \iota^* w_i(\gamma_1) = 0$ for $i > 1$ and the inclusion $\iota\colon \mathbf{R}P^n \longrightarrow G_1$, so we just need to worry about w_1. Consider the inclusion $j\colon \mathbf{R}P^1 \longrightarrow \mathbf{R}P^n$. We have $j^* w_1(\mathbf{R}P^n) = w_1(\mathbf{R}P^1) \neq 0$, as we proved before. Thus $w_1(\mathbf{R}P^n) \neq 0$, so it must be x.

From this we can conclude that there is no bundle E such that $\gamma_1 \oplus E$ is trivial. Indeed, if such a bundle existed we must have $w_i(E) = x^i$ for all i—which means that the bundle is infinite-dimensional.

Example 6.11. Let $\gamma_{1n}^\perp$ be the orthogonal complement to γ_{1n} in $\mathbf{R}^{n+1}$. Then $w_i(\gamma_{1n}^\perp) = x^i$. We can prove this by induction by noting that $\gamma_{1n} \oplus \gamma_{1n}^\perp = \epsilon^{n+1}$.

Remark 6.12. The previous example is one reason that the abomination is such a prevalent object. If we were to speak of it in those terms, we could simply write that $w(\gamma_{1n})w(\gamma_{1n}^\perp) = 1$ and note that $w(\gamma_{1n}) = 1 + x$ and

$$(1+x)(1+x+\cdots+x^n) = 1 \in \mathbb{Z}/2[x]/(x^{n+1}).$$

Our next example is an interesting exploration of how vector bundles can become simpler when trivial bundles are added in.

Example 6.13. Let $T\mathbf{R}P^n$ be the tangent bundle to $\mathbf{R}P^n$. We have a map $S^n \longrightarrow \mathbf{R}P^n$ sending x to $\{\pm x\}$. We can identify $T\mathbf{R}p^n$ with the set of pairs $\{(x,v), (-x,-v)\}$ with $x \in S^n$ and v tangent to S^n at x. Such a pair uniquely determines a linear mapping $\ell\colon \mathbf{R}x \longrightarrow (\mathbf{R}x)^\perp$, where $\ell(x) = v$. Thus we can canonically identify the fiber above x with the vector space $\mathrm{Hom}((\mathbf{R}x), (\mathbf{R}x)^\perp)$, and this lifts to an identification of $T\mathbf{R}P^n$ with $\mathrm{Hom}(\gamma_{1n}, \gamma_{1n}^\perp)$.

This doesn't allow us to compute the Steifel–Whitney classes of $T\mathbf{R}P^n$, since we don't know how to compute Steifel–Whitney classes of Hom-bundles. However, it turns out that by adding a trivial bundle we can turn this into something which lets us compute the classes.

Consider the sum $T\mathbf{R}P^n \oplus \epsilon^1$. We can write $\epsilon^1 \cong \mathrm{Hom}(\gamma_{1n}, \gamma_{1n})$, since the latter is a line bundle with an everywhere-nonzero section. Thus

$$T\mathbf{R}P^n \oplus \epsilon^1 \cong \mathrm{Hom}(\gamma_{1n}, \gamma_{1n}^\perp) \oplus \mathrm{Hom}(\gamma_{1n}, \gamma_{1n}) \cong \mathrm{Hom}(\gamma_{1n}, \gamma_{1n} \oplus \gamma_{1n}^\perp) \cong \mathrm{Hom}(\gamma_{1n}, \epsilon^{n+1}).$$

But this can be rewritten as $\mathrm{Hom}(\gamma_{1n}, \epsilon^1)^{\oplus n+1}$. However, $\mathrm{Hom}(\gamma_{1n}, \epsilon^1) \cong \gamma_{1n}$, so

$$T\mathbf{R}P^n \oplus \epsilon^1 \cong \gamma_{1n}^{\oplus n+1}.$$

Since adding a trivial bundle does not change Steifel–Whitney classes, we can use induction (or the abomination) to conclude that

$$w_i(T\mathbf{R}P^n) = \binom{n+1}{i} x^i \pmod{2}.$$