7. Geometric consequences

Example 6.12 above has enough interesting geometric consequences that are consequential enough that they are worth discussing in detail. We follow [MS74, Section 4].

Definition 7.1. A manifold is *parallelizable* if its tangent bundle is trivial.

Proposition 7.2. $\mathbf{R}P^n$ *is parallelizable only if* $n = 2^k - 1$ *for some* k.

Proof. If M is parallizable then $w_i(M) = 0$ for all $i > 0$. From Example 6.12 we know that $w_i(M) = \binom{n+1}{i} x^i$; these are all 0 exactly when $n + 1 = 2^k$ for some k. $\square$

In fact, the parallelizable real projective spaces are exactly $\mathbf{R}P^1$, $\mathbf{R}P^3$ and $\mathbf{R}P^7$; we will prove that these *are* parallelizable in a minute, and we will show that the rest are not later when we introduce K-theory.

The question of parallelizability of projective space is closely related to the question of division algebra structures on $\mathbf{R}^n$.

Theorem 7.3 (Steifel)**.** *Suppose that there exists a bilinear product operation* $p\colon \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *without zero divisors. Then the projective space* $\mathbf{R}P^{n-1}$ *is parallelizable.*

Note that p is not required to be associative or to have a unit.

Proof. Our goal is to construct a trivialization of $T\mathbf{R}P^{n-1}$ using p. Thus we want to construct $n-1$ linearly independent sections of $T\mathbf{R}P^{n-1} \cong \mathrm{Hom}(\gamma_{1(n-1)}, \gamma_{1(n-1)}^{\perp})$.

Note that given a linear map $T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ we can construct a map $\overline{T}\colon \ell \longrightarrow \ell^{\perp}$ for any line ℓ through the origin by taking $x \in \ell$ to the orthogonal projection of $T(x)$ onto $\ell^{\perp}$. Then $\overline{T}$ is a section of $T\mathbf{R}P^{n-1}$.

Now suppose that $T_1, \ldots, T_n\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ are n linear transformations such that $T_1(x), \ldots, T_n(x)$ are linearly independent for all x and such that $T_1 = 1$. Then for all x, $\overline{T}_2(x), \ldots, \overline{T}_n(x)$ are everywhere linearly independent, and thus give $n-1$ linearly independent sections of $T\mathbf{R}P^{n-1}$.

It thus remains to construct $T_1, \ldots, T_n$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{R}^n$, and consider the map $S = p(-, e_1)$. This is a linear map $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ with trivial kernel (since there are no zero divisors), so it is an isomorphism. We let

$$T_i(z) = p(S^{-1}(z), e_i).$$

Then $T_1 = 1$ and $T_1(x), \ldots, T_n(x)$ are linearly independent for all x, as desired. $\square$

Putting this together with Proposition 7.2, we see that $\mathbf{R}^n$ can be given a division algebra structure only when n is a power of 2. When $n = 1, 2, 4, 8$ such structures exist: they are the real, complex, quaternionic and octonic structures. When we prove that $\mathbf{R}P^n$ is not parallelizable for $n > 7$ we will simultaneously show that these are the only division algebra structures on $\mathbf{R}^n$.

For the last application of this section, we turn to a geometric consideration: when can a manifold M be immersed into $\mathbf{R}P^n$? (Recall that an immersion, unlike an embedding, can self-intersect, but only transversely.)

Suppose that an n-manifold M is immersed into $\mathbf{R}^{n+k}$. Let ν be the normal bundle, so that $\nu \oplus TM = \epsilon^{n+k}$. Thus we have $\overline{w}_i(TM) = w_i(\nu)$; since ν is k-dimensional we must have $\overline{w}_i(TM) = 0$ for $i > k$.

For example, when $M = \mathbf{R}P^9$ we have $w_i(TM) = x^i$ exactly when $i = 2, 8$, and thus $\overline{w}_i(TM) = x^i$ exactly when $i = 2, 4, 6$. Therefore if $\mathbf{R}P^9$ can be immersed into $\mathbf{R}^{9+k}$ we must have $k \geq 6$. Note that when $M = \mathbf{R}P^{2^k}$ we have $w_i(TM) = x^i$ exactly when $i = 1, n$; thus $\overline{w}_i(TM) = x^i$ exactly when $i = 1, 2, \ldots, 2^k - 1$. In particular, $\mathbf{R}P^{2^k}$ cannot be immersed in $\mathbf{R}^{2^k+\ell}$ unless $\ell \geq 2^k - 1$.

On the other hand, Whitney's immersion theorem [Whi44] states that any n-manifold can be immersed in $\mathbf{R}^{2n-1}$. We have thus shown that this bound is sharp: there is no $k > 1$ such that an n-manifold can always be immersed in $\mathbf{R}^{2n-k}$.