8. Stiefel–Whitney numbers

Further reading: [MS74, Section 4] for first part

We now consider a structure which is coarser than Steifel–Whitney classes, but which is still surprisingly powerful. These are the *Steifel–Whitney numbers.*

Definition 8.1. Let M be an n-manifold, and let $[M] \in H_n(M)$ be its fundamental class. Let $r_1, \dots, r_n$ be numbers such that

$$r_1 + 2r_2 + \cdots + nr_n = n.$$

Then the $(r_1, \dots, r_n)$-Steifel–Whitney number is

$$(w_1(TM)^{r_1} w_2(TM)^{r_2} \cdots w_n(TM)^{r_n})[M] \in \mathbb{Z}/2.$$

This is generally denoted

$$w_1^{r_1} \cdots w_n^{r_n}[M].$$

The monomial in cohomology is in degree n, so it is exactly a map $H_n(M) \longrightarrow \mathbb{Z}/2$ and thus can be applied to the fundamental class.

As a quick exercise with the definition, we observe the following property:

Lemma 8.2. *Let M and N be two n-manifolds. For any $r_1, \dots, r_n$ with $r_1 + 2r_2 + \cdots + nr_n = n$, we have*

$$w_i^{r_1} \cdots w_n^{r_n}[M \amalg N] = w_1^{r_1} \cdots w_n^{r_n}[M] + w_1^{r_1} \cdots w_n^{r_n}[N].$$

Proof. Note that the inclusions $M \longrightarrow M \amalg N$ and $N \longrightarrow M \amalg N$ give isomorphisms

$$H^n(M \amalg N) \cong H^n(M) \times H^n(N) \qquad \text{and} \qquad H_n(M \amalg N) \cong H_n(M) \times H_n(N).$$

The tangent bundles on M and N can be obtained as pullbacks of the tangent bundle on $M \amalg N$ via pullbacks along these inclusions. The image of $[M \amalg N]$ is exactly $([M], [N])$. The statement of the lemma follows from the fact that Steifel–Whitney classes commute with pulling back vector bundles. □

We now compute an example.

Example 8.3. Let us compute these for projective spaces. First, suppose that n is even. Then $w_n(T\mathbf{R}P^n) = (n+1)x^n \neq 0$, and thus $w_n[M] \neq 0$. Similarly, since $w_1(TM) = x$, $w_1^n[M] \neq 0$. In genereal, we have

$$w_1^{r_1} \cdots w_n^{r^n}[M] = \binom{n+1}{r_1}\binom{n+1}{r_2} \cdots \binom{n+1}{r_n} \bmod 2.$$

Depending on $n+1$, these vary. For example, when $n = 2^k - 2$ all of these are nonzero; on the other hand, when $n = 2^k$ the only ones which are nonzero are $w_1^n[M]$ and $w_n[M]$.

Now suppose that $n = 2k - 1$ is odd. Note that $(1+x)^{2k} = (1+x^2)^k$, and thus $\binom{2k}{2i} = \binom{k}{i}$ (mod 2) and $\binom{2k}{2i+1} = 0$ (mod 2). Thus in particular all odd Steifel–Whitney classes are zero. Since any $w_1^{r_1} \cdots w_n^{r_n}$ must have at least one $r_i \neq 0$ for an odd i, we see that all Steifel–Whitney numbers are 0.

From this example we can see that Steifel–Whitney numbers contain much less information than the classes themselves. However, it turns out that they contain just enough information to classify when a manifold is a boundary of another manifold.

Theorem 8.4 (Pontrjagin)**.** *If B is a smooth compact $(n+1)$-manifold with boundary M then all Steifel–Whitney numbers of M are* 0*.*

Proof. Consider the long exact sequence in homology for the pair (B, M). We have

$$H_{n+1}(M) \longrightarrow H_{n+1}(B) \longrightarrow H_{n+1}(B, M) \xrightarrow{\partial} H_n(M) \xrightarrow{i_*} H_n(B).$$

The first of these is 0. Note that $i_*[M]$ is the homology class in B represented by M; this is zero because M is a boundary: the boundary of B. Thus $[M]$ is in the image of a class in $[B, M] \in H_{n+1}(B, M)$, which we also consider to be the fundamental class. For any $v \in H^n(M)$ we have

$$v[M] = v(\partial[B, M]) = (\delta v)[B, M],$$

where $\delta\colon H^n(M) \longrightarrow H^{n+1}(B, M)$.

Let TB be the tangent bundle to B. Restricting it to M, we note that it has an everywhere-nonzero section, taking each point of M to the outward-facing vector. The orthogonal complement to this is the tangent bundle to M, so

$$TB|_M \cong TM \cong \epsilon^1.$$

Thus the Steifel–Whitney classes of $TB|_M$ are the same as those of TM. If we thus consider the exact sequence in cohomology

$$H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)$$

we see that $w_k TM = i^* w_k TB|_M$ for all k. Thus

$$w_1^{r_1} \cdots w_n^{r_n}[M] = w_1^{r_1} \cdots w_n^{r_n}(\partial[B, M]) = (\delta(w_1^{r_1} \cdots w_n^{r_n}))[B, M] = (\delta i^*(w_1^{r_1} \cdots w_n^{r_n}))[B, M] = 0.$$

Thus all Steifel–Whitney numbers of M are 0, as desired. □

Thus the Steifel–Whitney numbers can identify exactly when a manifold is a boundary. This leads to an interesting corollary for cobordisms:

Definition 8.5. Let M and N be two n-manifolds. M and N are *cobordant* if there exists an $(n+1)$-manifold W such that $\partial W = M \amalg N$. W is generally referred to as the *cobordism* between M and N.

This definition is the definition of *unoriented cobordism*. There are many other cobordism types, including oriented cobordism, complex cobordism, framed cobordism, etc: all of these take manifolds with some sort of structure and require the cobordism to carry this structure as well. Thus, for example, with oriented cobordism we require an orientation on W that restricts to the correct orientations on M and N. In this class we focus on unoriented cobordism; if we have time, we may revisit other types of cobordism later in the term.

Example 8.6. Any manifold is cobordant to itself, since the boundary of $M \times I$ is exactly $M \amalg M$.

Example 8.7. A circle S^1 is cobordant to $S^1 \amalg S^1$ via the *pair of pants*. In fact, it is a reasonably good guess that the reader is wearing both a cobordism between S^1 and $S^1 \amalg S^1$ (the reader's pants) and a cobordism between S^1 and $S^1 \amalg S^1 \amalg S^1$ (the reader's shirt).

Example 8.8. Cobordisms are not unique. $S^1 \times I$ is a cobordism between S^1 and S^1, but so is the torus with two ends cut off. In fact, the structure of cobordisms is quite interesting; it is addressed in the famous cobordism theorem, such as the h-cobordism theorem and the s-cobordism theorem. See for example [Mil65].

Corollary 8.9. *If M and N are cobordant then their Steifel–Whitney numbers are equal.*

Proof. This follows directly from the theorem and Lemma 8.2. □

The theorem is a special case of the corollary when $N = \emptyset$; what the proof of the corollary shows is that the special case contains all of the complexity of the general case.

A natural question is whether the theorem is an if and only if: given two manifolds M and N with equal Steifel–Whitney numbers, are M and N cobordant? In [Tho54] Thom proves that the answer is yes. In fact, he proves even more by identifying the structure of the cobordism groups.

Definition 8.10. The *unoriented cobordism group* $\mathfrak{N}_n$ is defined as follows. As a set, $\mathfrak{N}_n$ consists of the equivalence classes of n-manifolds up to cobordism. Addition is defined to be $\amalg$.

Lemma 8.11. *$\mathfrak{N}_n$ is an abelian group.*

Proof. First we check that the operation is well-defined. Suppose that $[M] = [M]'$ and $[N] = [N']$. We need to check that $[M \amalg N] = [M' \amalg N']$. Let W be a cobordism between M and M' and W' a cobordism between N and N'. Then $W \amalg W'$ is a cobordism between $M \amalg N$ and $M' \amalg N'$, so the operation is well-defined.

The identity is $[\emptyset]$. (If $\emptyset$ is not considered an n-manifold, we can consider $[S^n]$, since S^n is always a boundary.) Note that $2[M] = 0$ for all M, since $\partial(M \times I) = [M \amalg M]$. Thus inverses exist in the group.

$\mathfrak{N}_n$ is abelian because $M \amalg N \cong N \amalg M$. ☐

We will spend the next few lectures proving Thom's theorems. As before, we will rely heavily on the Thom isomorphism theorem.