9. More on Thom spaces

Further reading: Thom's paper [Tho54] contains a *lot* of very cool material. Unfortunately, given that this was the original paper for much of this material, the notation is very nonstandard and it makes it somewhat difficult to read. The original paper is in French, but is well-worth the read. There is an English translation in [NT07]. However, this is a translation of the Russian translation of the original, and thus has some mistakes that the original did not.

In this lecture we will spend a bit more time on Thom spaces than we did before, as we will need this theory to classify cobordism classes. Previously, all we needed was the Thom isomorphism theorem, but now we actually want to understand the geometry of Thom spaces better.

Lemma 9.1. *For any base B, $\mathrm{Th}(\epsilon^k) \cong S^k \wedge B_+$.*

Proof. Consider $D(\epsilon^k) \cong D^k \times B$. When we take the Thom space we quotient out by $S^{k-1} \times B$. If we first quotient by S^{k-1} on each boundary, we obtain the space $S^k \times B$, with S^k pointed (by the point which is the image of the boundary of D^k). We then collapse $* \times B$ to a single point to obtain $\mathrm{Th}(\epsilon^k)$. The space $S^k \wedge B_+$ is exactly this same quotient, so they are isomorphic. □

Lemma 9.2. *For any bundles $E \longrightarrow B$ and $E \longrightarrow B'$, we have*

$$\mathrm{Th}(E \times E') \cong \mathrm{Th}(E) \wedge \mathrm{Th}(E').$$

Proof. Note that $\mathrm{Th}(E \times E')$ is the one-point compactification of $E \times E'$. But this one-point compactification is exactly the smash product of the one-point compactification of E and the one-point compactification of E', which is exactly $\mathrm{Th}(E) \wedge \mathrm{Th}(E')$, as desired. □

Corollary 9.3.

$$\mathrm{Th}(E \oplus \epsilon^k) \cong \mathrm{Th}(E) \wedge S^k.$$

Proof. Note that $E \oplus \epsilon^k \cong E \times \epsilon^k$, where in the right-hand side we think of ϵ^k as a bundle over the point. Thus

$$\mathrm{Th}(E \oplus \epsilon^k) \cong \mathrm{Th}(E) \wedge \mathrm{Th}(\epsilon^k).$$

But $\mathrm{Th}(\epsilon^k)$ is the one-point compactification of $\mathbf{R}^k$, which is exactly S^k. □

We can use this corollary to construct the *Thom spectrum* of a vector bundle. For any bundle $E \longrightarrow B$, we can define a spectrum

$$\mathrm{Th}(E), \mathrm{Th}(E \oplus \epsilon^1), \mathrm{Th}(E \oplus \epsilon^2), \ldots$$

The structure maps of this spectrum are given by the isomorphisms

$$\Sigma\mathrm{Th}(E \oplus \epsilon^k) \cong S^1 \wedge \mathrm{Th}(E \oplus \epsilon^k) \cong \mathrm{Th}(\epsilon^1 \oplus E \oplus \epsilon^k) \cong \mathrm{Th}(E \oplus \epsilon^{k+1}).$$

Lemma 9.4. *For $k > n$, the group $\pi_{n+k}(\mathrm{Th}(\gamma_k))$ is independent of k.*

Note that we know this is true for cohomology groups: By the Thom isomorphism theorem, we know that $H^{n+k}(\mathrm{Th}(\gamma_k)) \cong H^n(G_k)$. Since $n < k$, this is going to be the group

$$\mathbb{Z}/2\{w_1^{i_1} \cdots w_n^{i_n} \mid i_1 + 2i_2 + \cdots + ni_n = n\}.$$

This is clearly independent of k. The point of this lemma is that this is also true for homotopy groups.

In the proof of this lemma we will need two theorems, which we state here without proof; their proofs can be found in [Hat02].

Theorem 9.5 (Freudenthal Suspention Theorem)**.** *If X is an n-connected CW complex then the suspension homomorphism*

$$\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i < 2n$.

Theorem 9.6 (Hurewicz)**.** *For any space X and any positive integer k there exists a homomorphism $h\colon \pi_i X \longrightarrow H_i X$ defind by*

$$(f\colon S^i \longrightarrow X) \longmapsto f_*[S^i].$$

When $i = 1$ this is the abelianization map if X. When X is $(n-1)$-connected for $n \geq 2$ this is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$.

This theorem also works relatively: given a

In addition, we need one more proposition which is a consequence of Hurewicz; this is [Tho54, Theorem II.6].

Proposition 9.7. *Let X and Y be simply connected CW complexes and let $f\colon X \longrightarrow Y$. Suppose that for all primes p, the induced map $f^*\colon H^i(Y;\mathbb{Z}/p) \longrightarrow H^i(X;\mathbb{Z}/p)$ is an isomorphism when $i < k$ and injective for $i = k$. Then there exists a map $g\colon Y^{(n)} \longrightarrow X^{(n)}$ such that $f \circ g$ and $g \circ f$ are homotopy inverses on $n-1$-skeletons.*

We are now ready to prove the lemma.

Proof of Lemma 9.4. Recall the map that induces η on cohomology groups:

$$G_k \xleftarrow{\sim} S(\gamma_{k+1}) \hookrightarrow D(\gamma_{k=1}) \xrightarrow{\sim} G_{k+1}.$$

Since both G_k and $S(\gamma_{k+1}x)$ are CW complexes, the first of these has a homotopy inverse, and we can consider a map $\iota\colon G_k \longrightarrow G_{k+1}$. This map induces a map on Thom spaces $i\colon \mathrm{Th}(\gamma_k) \longrightarrow \mathrm{Th}(\gamma_{k+1})$. We want to show that for $n < k$, the induced map

$$\pi_{n+k}\mathrm{Th}(\gamma_k) \xrightarrow{i_*} \pi_{n+k+1}\mathrm{Th}(\gamma_{k+1})$$

is an isomorphism.

First, consider the Thom space $\mathrm{Th}(\gamma_k)$. By the Thom isomorphism theorem we know that $H^n(\mathrm{Th}(\gamma_k)) \cong H^{n-k}(G_k)$, so for $n < k$ this is 0. In addition,

$$\pi_1\mathrm{Th}(\gamma_k) = \pi_1(D(\gamma_k))/\operatorname{im}\pi_1(S(\gamma_k)) \cong \pi_1(G_k)/\operatorname{im}\pi_1(G_{k-1}),$$

where the image of $\pi_1(G_{k-1})$ is the image under the inclusion $S(\gamma_k) \hookrightarrow D(\gamma_k)$; note that it must therefore also be the image under i. Now $\pi_1(G_k) \cong \mathbb{Z}/2$, since $G_k = BO(k)$ and $\pi_1 BO(k) \cong \pi_0 O(k) = \mathbb{Z}/2$.[‖] To check that $\pi_1(G_{k-1}) \longrightarrow \pi_1(G_k)$ is onto it suffices to note that this is the map $BO(k-1) \longrightarrow BO(k)$ which corresponds to adding a 1 in the lower-right corner of the matrix. This map is clearly onto on π_0, and thus applying B to it will make it onto on π_1. Thus $\mathrm{Th}(\gamma_k)$ is simply connected. Then by Hurewicz we know that $\pi_n\mathrm{Th}(\gamma_k) = 0$ for $n < k$. Thus $\mathrm{Th}(\gamma_k)$ is $k-1$-connected.

By the Freudenthal suspension theorem, we then have that the map

$$\pi_{n+k}(\mathrm{Th}(\gamma_k)) \longrightarrow \pi_{n+k+1}(S^1 \wedge \mathrm{Th}(\gamma_k))$$

is an isomorphism for $n \leq k$. From above, we know that

$$S^1 \wedge \mathrm{Th}(\gamma_k) \cong \mathrm{Th}(\gamma_k \oplus \epsilon^1).$$

The bundle $\gamma_k \oplus \epsilon^1$ is the pullback of γ_{k+1} along i. Thus i induces a map $\mathrm{Th}(\gamma_k \oplus \epsilon^1) \longrightarrow \mathrm{Th}(\gamma_{k+1})$. On cohomology this map is an isomorphism for $n \leq k$.

We want to show that $\pi_{n+k+1}\mathrm{Th}(\gamma_k \oplus \epsilon^1) \longrightarrow \pi_{n+k+1}\mathrm{Th}(\gamma_{k+1})$ is an isomorphism for $n < k$. We know that it is an isomorphism on cohomology for $n \leq k$, and that both of these spaces are simply connected. Thus we can apply Proposition 9.7 for the desired conclusion. □

This is the first occurrence of *stable homotopy groups.*

[‖] This follows from the long exact sequence in homotopy for a fiber bundle.