MATH 3110: HOMEWORK 11

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 0 (don't submit any of this for a grade)**.**

(i) Prove that a uniform limit of bounded functions is bounded.
(ii) Verify that uniform convergence implies pointwise convergence.
(iii) Do Exercise 6.2.6 in the textbook.
(iv) Use the Fundamental Theorem of Calculus and Darboux's Theorem to give a new proof of the Intermediate Value Theorem.

Problem 1. Suppose that f is continuous on $[0,1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\,dx = f(0).$$

Problem 2. Say that a set $X \subseteq [a,b]$ has *content zero* if for every $\varepsilon > 0$ there are finitely many intervals $[a_1,b_1], [a_2,b_2], \ldots, [a_n,b_n]$ such that $X \subseteq \bigcup_{k\leq n}[a_k,b_k]$ and $\sum_{1\leq k\leq n} b_k - a_k < \varepsilon$. (The first condition says that the intervals cover X; the second says that their total length is small.)

(a) Show that every finite set has content zero.
(b) Show that the Cantor set $\mathcal{C}$ has content zero. (But remember that it is uncountable.)
(c) Suppose that f is a bounded function on $[a,b]$ and that the set of points $x \in [a,b]$ at which f is discontinuous has content zero. Prove that f is integrable.
(d) Prove that the function

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{if } x \notin \mathcal{C} \end{cases}$$

is integrable and compute $\int_0^1 h$. (Here $\mathcal{C}$ is the Cantor set.)

Problem 3. Suppose that $f\colon [a,b] \to [0,\infty)$ is a continuous function with maximum value 2. Prove that

$$\lim_{n\to\infty} \left(\int_a^b f^n \right)^{1/n} = 2.$$

Date: Due Wednesday, 1 May 2019.

Problem 4. Integration by parts! Suppose that u and v are functions $[a, b] \to \mathbf{R}$ with continuous derivatives. Prove that

$$\int_a^b uv' = u(b)v(b) - u(a)v(a) - \int_a^b u'v.$$

Problem 5. Prove that a sequence of functions has a uniform limit if and only if it is uniformly Cauchy.

Problem 6 (textbook 6.2.7)**.** Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is uniformly continuous, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that (f_n) converges uniformly to f.

Give an example to show that this fails if we assume only that f is continuous and not uniformly continuous.

Bonus problem[1]**.** Suppose that $f\colon [0, 1] \to \mathbf{R}$ is integrable. Prove that there is a point $c \in (0, 1)$ at which f is continuous.

[1] Don't work on this until you've solved the other problems. If you do solve it, submit your solution directly to me (Zach), not to Gradescope.