MATH 3110: HOMEWORK 2

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 1. (First — not on your homework, just for yourself — write down the *definition* of the absolute value function.)

(a) Prove the following formula for the maximum of two real numbers x and y:

$$\max(x, y) = \tfrac{1}{2}(x + y + |x - y|).$$

(b) Prove that the equality $|ab| = |a| \cdot |b|$ holds for all real numbers a and b.

(c) Prove, by first establishing the inequality $(a + b)^2 \leq (|a| + |b|)^2$ or otherwise, the *triangle inequality*:[1]

$$|a + b| \leq |a| + |b|.$$

(d) Prove that $|a - c| \leq |a - b| + |b - c|$ for all $a, b, c \in \mathbf{R}$. (*Hint*: Use (c).)

Problem 2. Recall that you showed $\sqrt{3}$ to be irrational. The same proof shows that $\sqrt{5}$ is irrational. By considering their product or otherwise, prove that $\sqrt{3} - \sqrt{5}$ and $\sqrt{3} + \sqrt{5}$ are either both rational or both irrational. Deduce that they must both be irrational.

Problem 3. For each item, compute the requested supremum or infimum or carefully explain why it does not exist. Either way, *prove* that your answer is correct.

(a) Determine $\sup A$ for $A = \left\{ \frac{(-1)^n}{n} : n \in \mathbf{N} \setminus \{0\} \right\}$.

(b) Fix $\alpha \in (0, 1)$. Determine $\inf(B)$ for $B = \{\alpha^n : n \in \mathbb{N}\}$.

(c) Fix $\alpha \in (1, \infty)$. Determine $\sup(C)$ for $C = \{\alpha^n : n \in \mathbb{N}\}$.

Problem 4. Prove each of the following assertions.

(a) If x is an upper bound for a set $S \subseteq \mathbf{R}$ and x is also a member of S, then $x = \sup S$.

(b) Suppose that x is an upper bound for a set $S \subseteq \mathbf{R}$. Then $x = \sup S$ if and only if for every $n \in \mathbf{N}$ there is an element of S in the interval $(x - \frac{1}{n}, x]$.

(c) If $A \subseteq B$ are nonempty, bounded sets of real numbers, then

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Problem 5. (a) Suppose that $A, B \subseteq \mathbf{R}$ are nonempty and bounded above. Find a formula for $\sup(A \cup B)$ and prove that it is correct.

(b) Suppose that $\alpha > 0$ and that $A \subseteq \mathbf{R}$ is nonempty and bounded above. Let $C = \{\alpha x : x \in A\}$. Prove that C is nonempty and bounded above and that $\sup C = \alpha \sup A$.

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[1] Meditate on why it is called this.

(c) Suppose that $A, B \subseteq \mathbf{R}$ are both nonempty and bounded above. Let

$$C = \{a + b : a \in A,\ b \in B\}.$$

Prove that C is nonempty and bounded above and that $\sup C = \sup A + \sup B$.

Problem 6. Give an example satisfying the requested condition or prove that no such example can exist.

(a) A sequence all of whose terms are (strictly) negative that converges to 0.
(b) A sequence all of whose terms are (strictly) negative that converges to 0.01.
(c) A sequence with infinitely many 0s that does not converge to 0.
(d) A sequence with infinitely many 0s that converges to a number $\neq 0$.
(e) A sequence all of whose terms are non-integers that converges to an integer.
(f) A sequence all of whose terms are integers that converges to a number that is not an integer.
(g) Two non-convergent sequences (a_n) and (b_n) such that the sequence $(a_n + b_n)$ converges.