MATH 3110: HOMEWORK 3

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 1. Suppose that for each n we have a closed interval $I_n = [a_n, b_n]$ and that the intervals are decreasing: $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$. Prove that there is a real number that belongs to all of the intervals: $\bigcap_{n\in\mathbf{N}} I_n \neq \varnothing$.

Problem 2. Suppose that X is a nonempty set. Prove that the following three assertions are equivalent:

(a) X is finite or countably infinite.
(b) There is a one-to-one function $f\colon X \to \mathbf{N}$.
(c) There is an onto function $g\colon \mathbf{N} \to X$.

(*Hint*: You might find it easiest to prove (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c).)

Problem 3. This problem shows that "equinumerosity is an equivalence relation." (This justifies the notation $|A| = |B|$.) Let A, B, and C be sets. For this problem only, we'll write $A \sim B$ to mean that A and B are equinumerous, meaning that there is a bijection $A \to B$.

(a) Show that $A \sim A$.
(b) Show that if $A \sim B$ then $B \sim A$.
(c) Show that if $A \sim B$ and $B \sim C$, then $A \sim C$.

Problem 4. A set $I \subseteq \mathbf{R}$ is called an **interval** if for all triples of real numbers x, y, and z such that $x < z < y$, if $x \in I$ and $y \in I$ then $z \in I$.

(a) Prove that every interval takes exactly one of the following nine forms:

$$(-\infty, a),\ (-\infty, a],\ (a, b),\ (a, b],\ [a, b),\ [a, b],\ (a, \infty),\ [a, \infty),\ \text{or } (-\infty, \infty).$$

(b) Give an example of an infinite family of intervals of the form (a, b) of which any two are disjoint.
(c) Show that part (b) *cannot* be solved with an uncountable family of intervals.

Problem 5. Suppose that $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are sequences and that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M \neq 0$. Prove that $\lim_{n\to\infty} c_n = L/M$, where

$$c_n = \begin{cases} \frac{a_n}{b_n} & \text{if } b_n \neq 0 \\ -3.17 & \text{if } b_n = 0. \end{cases}$$

Date: Due Wednesday, 13 February 2019.

(*Hint:* Start by showing that the -3.17 case occurs infrequently.)

Problem 6. Suppose that $(a_n)_{n\in\mathbf{N}}$ is a sequence of nonnegative real numbers that converges to L. Show that the sequence $(\sqrt{a_n})_{n\in\mathbf{N}}$ converges to $\sqrt{L}$.

Problem 7 (The Sandwich Theorem for sequences)**.** Suppose that $(l_n)_{n\in\mathbf{N}}$, $(u_n)_{n\in\mathbf{N}}$, and $(c_n)_{n\in\mathbf{N}}$ are all sequences such that $l_n \le c_n \le u_n$ for every $n \in \mathbf{N}$. Suppose also that $\lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n = L$. Prove that $(c_n)_{n\in\mathbf{N}}$ also converges to L.[1]

[1] l is for *lower* and u is for *upper*.