# MATH 3110: HOMEWORK 4

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

**Problem 1.** I'm thinking of ten different ten-digit numbers (e.g. 4860239777). If you choose a ten-digit number that's *different* from all ten of my numbers, you win a prize. You can ask me ten yes-or-no questions about my numbers. Which ten questions do you ask, and how do you use the answers to *guarantee* that you win the prize? (*Hint:* This is related to what we've proved in class recently.)

**Problem 2.** For each item, provide an example (and prove that it works) or prove that no such example exists.

(a) A bounded sequence that does not converge to $\frac{4}{9}$ but has a subsequence converging to $\frac{4}{9}$.
(b) A monotone sequence that does not converge to $\frac{4}{9}$ but has a subsequence converging to $\frac{4}{9}$.
(c) A sequence with both an increasing subsequence and a decreasing subsequence that does not converge.
(d) A bounded monotone sequence that does not converge.
(e) A sequence that does not converge and has no convergent subsequences.
(f) A bounded sequence with an unbounded subsequence.

**Problem 3.** For each item, provide an example (and prove that it works) or prove that no such example exists.

(a) A sequence $(a_n)_{n\in\mathbf{N}}$ such that $3 < a_n < 4$ for all $n \in \mathbf{N}$ and $(a_n)_{n\in\mathbf{N}}$ has a subsequence converging to 3 and another subsequence converging to 4.
(b) A sequence $(a_n)_{n\in\mathbf{N}}$ such that, for each $k \in \mathbf{N} \setminus \{0\}$, there is a subsequence of $(a_n)_{n\in\mathbf{N}}$ converging to $\frac{1}{k}$.
(c) A sequence $(a_n)_{n\in\mathbf{N}}$ such that, for each $k \in \mathbf{N} \setminus \{0\}$, there is a subsequence of $(a_n)_{n\in\mathbf{N}}$ converging to $\frac{1}{k}$, but there is no subsequence of $(a_n)_{n\in\mathbf{N}}$ converging to 0.
(d) A sequence $(a_n)_{n\in\mathbf{N}}$ such that for every real number $x$, the sequence $(a_n)_{n\in\mathbf{N}}$ has a subsequence that converges to $x$.

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*Date*: Due Wednesday, 20 February 2019.

**Problem 4.** Suppose that $(a_n)_{n\in\mathbf{N}}$ is a decreasing sequence of nonnegative real numbers. Prove that the series $\sum_{n\geq 0} a_n$ converges if and only if the series

$$\sum_{n\geq 0} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

(*Hint:* To prove the implication $\Leftarrow$, imitate the proof that the harmonic series diverges.)

**Problem 5** (essentially Exercise 2.4.7 of the textbook)**.** Let $(a_n)_{n\in\mathbf{N}}$ be a bounded sequence, and define the auxiliary *limit superior* and *limit inferior* sequences as follows:

$$S_n = \sup\{a_k : k \geq n\} \quad \text{and} \quad I_n = \inf\{a_k : k \geq n\}.$$

(a) Show (in roughly one sentence, please) that $(S_n)_{n\in\mathbf{N}}$ is a decreasing sequence and that $(I_n)_{n\in\mathbf{N}}$ is an increasing sequence.

(b) Using the assumption that $(a_n)_{n\in\mathbf{N}}$ is bounded, prove that $(S_n)_{n\in\mathbf{N}}$ converges. Convince yourself (but don't submit this with your homework) that $(I_n)_{n\in\mathbf{N}}$ also converges.

We can therefore define the "limsup" and "liminf" of an arbitrary bounded sequence as follows:

$$\limsup a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\sup_{k\geq n} a_k\right]$$

$$\liminf a_n = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left[\inf_{k\geq n} a_k\right].$$

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence $(a_n)_{n\in\mathbf{N}}$, and give an example of a sequence for which the inequality is strict.

(d) Prove that $\liminf a_n = \limsup a_n$ if and only if $(a_n)_{n\in\mathbf{N}}$ converges. In this case, $\liminf a_n = \lim a_n = \limsup a_n$.