MATH 3110: HOMEWORK 5

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 1. Let $r \in (-1, 1)$. Show directly (without appealing to recent theorems from lecture) that the sequence of partial sums of the geometric series $\sum_{n\geq 0} r^n$ is Cauchy.

Problem 2.

- (✓) Show (but *do not* turn in with your homework) that changing finitely many terms of a series does not affect its convergence. That is, if $\sum a_n$ converges and there is $N \in \mathbf{N}$ such that $n \geq N$ implies $b_n = a_n$, then $\sum b_n$ converges too.
- (a) Prove that if $(na_n)_{n\in\mathbf{N}}$ converges to a nonzero real number L, then the series $\sum a_n$ diverges. (*Hint*: Comparison.) Give an example to show that the converse is false.
- (b) Prove that if $(n^2 a_n)_{n\in\mathbf{N}}$ converges (to any real number), then the series $\sum a_n$ converges. (*Hint*: Comparison.) Give an example to show that the converse is false.

Problem 3. Show how to construct a rearrangement of the alternating harmonic series that converges to 7.

Problem 4. The **Alternating Series Test** is the following fact: if $(a_n)_{n\in\mathbf{N}}$ is a decreasing sequence of nonnegative terms converging to 0, then the alternating series $\sum_{n\geq 0}(-1)^n a_n$ converges. By imitating our proof of the convergence of $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ or otherwise, prove the Alternating Series Test.

Problem 5. Prove that if $\sum_{n\geq 0} a_n$ converges absolutely to L, then any rearrangement of $\sum_{n\geq 0} a_n$ also converges to L.

Indicate clearly in your proof where you use the fact that the series is absolutely convergent.

(*Hint*: Consider the partial sums (s_n) of $\sum a_n$ and the partial sums (s'_n) of the rearrangement, and show that $\lim_{n\to\infty} \left|s_n - s'_n\right| = 0$.)

Problem 6. Prove the "p-test": the series $\sum_{n\geq 1} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

(*Hint*: Use Homework 4, Problem 4.)

Problem 7. Prove the Ratio Test: Suppose $(a_n)_{n\in\mathbf{N}}$ is a sequence converging to 0 and set $\rho = \limsup \left|\frac{a_{n+1}}{a_n}\right|$. If $\rho < 1$, then $\sum a_n$ is absolutely convergent.

(For the definition of limsup, see Homework 4, Problem 5.)

Date: Due Friday, 1 March 2019.