# MATH 3110: HOMEWORK 6

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

**Problem 1.** Recall the version of the Heine–Borel Theorem we proved in lecture:

**Theorem** (Heine–Borel)**.** Suppose that to each $x \in [a, b]$ is assigned a positive "jurisdiction radius" $\delta(x) > 0$. (That is, $\delta$ is a function $[a, b] \to (0, \infty)$.) There are $n \in \mathbf{N}$ and finitely many $x_1, \dots, x_n \in [a, b]$ such that

$$[a, b] \subseteq B_{\delta(x_1)}(x_1) \cup \cdots \cup B_{\delta(x_n)}(x_n). \tag{HB}$$

(a) When we prove something of the form "for all $x$ there is a $y$…" it is usually the case that the $y$ will depend on $x$. Show, by giving a family of examples, that the number $n$ in the conclusion of the Heine–Borel Theorem depends on $\delta$. More specifically, find for each $n \geq 1$ a function $\delta_n\colon [0, 1] \to (0, \infty)$ such that at least $n$ (and no fewer) elements of $[0, 1]$ are needed to satisfy (HB).

(b) Show that the Heine–Borel Theorem does not apply to the open interval $(0, 1)$. That is, construct a function $\delta\colon (0, 1) \to (0, \infty)$ such that no finitely many elements $x_1, \dots, x_n$ can be chosen from $(0, 1)$ to satisfy (HB).

(c) Show that the Heine–Borel Theorem does not apply to the set $[1, 2] \cap \mathbf{Q}$. That is, construct a function $\delta\colon [1, 2] \cap \mathbf{Q} \to (0, \infty)$ such that no finitely many elements $x_1, \dots, x_n$ can be chosen from $[1, 2] \cap \mathbf{Q}$ to satisfy (HB).

**Problem 2.** Suppose that $X$ is a bounded infinite set. Prove that $X$ has at least one cluster point. (*Hint*: Use your theorems.)

**Problem 3.** Show that there are *not* two nonempty disjoint open subsets of $\mathbf{R}$ whose union is all of $\mathbf{R}$.

**Problem 4.** Show that no set has uncountably many isolated points.
(*Hint*: Homework 3 #4(c) might be helpful.)

**Problem 5.**

(a) Recall from lecture that the Cantor set $\mathcal{C}$ was defined to be the intersection of sets $C_n$, where $C_n$ is a disjoint union of $2^n$ intervals, each of diameter $\frac{1}{3^n}$. For each $n \in \mathbf{N}$ let $E_n$ be the set of ($2^{n+1}$ many) endpoints of the intervals that compose $C_n$. (E.g. $E_2 = \{0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1\}$.) Notice that $\bigcup_{n \in \mathbf{N}} E_n$ is a countable subset of $\mathcal{C}$.[1] Show that the closure of $\bigcup_{n \in \mathbf{N}} E_n$ is $\mathcal{C}$.

(b) Let $\mathcal{C}$ be the Cantor set. Show that $\mathcal{C}$ is exactly the set of numbers in $[0, 1]$ whose ternary (i.e., base-3) expansion uses only 0s and 2s. That is, show that $x \in \mathcal{C}$ iff there is a sequence $(a_n)_{n \in \mathbf{N}}$ with $a_n \in \{0, 2\}$ for all $n$, such that $x = \sum_{n \geq 1} \frac{a_n}{3^n}$.

(c) Let $\mathcal{C}$ be the Cantor set. Prove (perhaps by using the previous part or by following the hints given in Exercise 3.3.7 in the textbook) that $\mathcal{C} + \mathcal{C} = [0, 2]$. (By $\mathcal{C} + \mathcal{C}$, I mean the set $\{x + y : x, y \in \mathcal{C}\}$.)

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*Date*: Due Wednesday, 13 March 2019.

[1] This follows from facts from lecture. You should prove it, but *don't* turn in your proof with your homework.