MATH 3110: HOMEWORK 8

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 0 (For extra practice — don't submit this one with your homework)**.**

(a) Prove that functional limits are unique. That is, if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$, then $L = M$.

(b) Prove that we can combine functional limits by e.g. addition and multiplication in the familiar way. (Corollary 4.2.4 of the textbook.)

(c) Prove the sequential characterization of functional limits. That is, prove that $\lim_{x\to c} f(x) = L$ iff for all sequences (x_n) with $x_n \in A \setminus \{c\}$ and $x_n \to c$, we have $f(x_n) \to L$.

(d) Prove that if $f\colon B \to \mathbf{R}$ is uniformly continuous on B and $A \subseteq B$, then f is uniformly continuous on A too. (By this we mean that the restriction $f \restriction A$ is uniformly continuous.)

(e) Prove that the function $x \mapsto x^2$ is *not* uniformly continuous on $[0, \infty)$.

Problem 1. Prove that if $c \in A$ is a cluster point of a set A, then $f\colon A \to \mathbf{R}$ is continuous at c iff $\lim_{x\to c} f(x) = f(c)$. (The familiar definition from introductory calculus.) What happens if c is not a cluster point of A?

Problem 2. Prove that a composite of continuous functions is continuous. That is, if $f\colon A \to \mathbf{R}$ and $g\colon B \to \mathbf{R}$ satisfy $f[A] \subseteq B$, then the composite function $g \circ f\colon A \to \mathbf{R}$ is continuous.

Problem 3 (Building familiar functions I)**.**

(a) For all $n \in \mathbf{N}$ prove that the function $p_n\colon [0, \infty) \to \mathbf{R}$ defined by $p_n(x) = x^n$ is strictly increasing and continuous.

(b) For all $n \in \mathbf{N}$ and $x \geq 0$ prove that there exists a unique $y \geq 0$ such that $y^n = x$. We denote this real number y by $x^{1/n}$ (or $\sqrt[n]{x}$).

(c) Prove that for all $n \in \mathbf{N}$ the function $x \mapsto x^{1/n}\colon [0, \infty) \to \mathbf{R}$ is increasing and continuous.

(Now we can *define* (!) $x^{p/q}$ to be $(x^p)^{1/q}$. So rational powers of real numbers are defined and behave as expected. Stay tuned for all real powers…)

Problem 4. Show that the function given by the formula $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Date: Due Friday (!), 29 March 2019.

Problem 5. Suppose that we have three real numbers $a < b < c$ and a function $g\colon (a, b) \to \mathbf{R}$ that is known to be uniformly continuous on the interval $(a, b]$ and on the interval $[b, c)$. Prove that g is uniformly continuous on (a, c).

Problem 6. Decide whether each of the following statements is true or false and prove that your answer is correct.

(a) If $f\colon [a, b] \to \mathbf{R}$ is a function with $f(x) > 0$ for all $x \in [a, b]$, then its reciprocal $1/f$ is a bounded function on $[a, b]$.

(b) If $f\colon [a, b] \to \mathbf{R}$ is a continuous function with $f(x) > 0$ for all $x \in [a, b]$, then its reciprocal $1/f$ is a bounded function on $[a, b]$.

(c) If $f\colon \mathbf{R} \to \mathbf{R}$ is a function with the property that for every compact set X, $f[X]$ is compact, then f must be continuous. (*Hint*: sequences)

Problem 7.

($\star$) (Don't submit this one) Review your solution to Homework 7, Problem 7, and make sure that you can prove: if $f\colon [a, b] \to \mathbf{R}$ is a continuous one-to-one function, then f is either strictly increasing or strictly decreasing.

(a) Prove that an increasing function $f\colon [a, b] \to \mathbf{R}$ with the *intermediate value property* (meaning, satisfying the conclusion of the Intermediate Value Theorem) must be continuous.

(b) Suppose that $f\colon [a, b] \to \mathbf{R}$ is a strictly increasing function with range R. Prove that f is one-to-one, so its inverse $f^{-1}\colon R \to [a, b]$ is defined. Prove also that f^{-1} is continuous.

(Notice that we have made no assumption about the continuity of f.)

Problem 8.

(a) Prove that there is no continuous onto function $g\colon [0, 1] \to (0, 1)$.

(b) Exhibit a continuous onto function $h\colon (0, 1) \to [0, 1]$.

(c) Prove that there is no continuous bijection $f\colon (0, 1) \to [0, 1]$.

(*Hint*: Use your theorems.)

Bonus problem[1]**.** Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is a function that is continuous at every rational number. Prove that there is an irrational number at which f is continuous.

[1]Don't work on this until you've solved the other problems. If you do solve it, submit your solution directly to me (Zach), not to Gradescope.