MATH 3110: PRACTICE FOR PRELIM 1

Basic topology of R. Since we haven't had a homework that covers open & closed sets, etc., I include some problems about that for you to practice.

Problem 1. For two sets A & B, we write $A + B = \{a + b : a \in A,\ b \in B\}$.

(a) Prove that if A and B are open, then $A + B$ is open.
(b) Prove that if A and B are compact, then $A + B$ is compact.
(c) *Caution*: It is not true that the sum of closed sets must be closed. Provide an example to demonstrate this.

Problem 2. Let $X \subseteq \mathbf{R}$. Let X' be the set of cluster points of X. Prove that if x is a cluster point of X', then x is a cluster point of X.

Problem 3. Let $\mathcal{C}$ be the Cantor set. Show that $\mathcal{C}$ is exactly the set of numbers in $[0,1]$ whose ternary (i.e., base-3) expansion uses only 0s and 2s. That is, show that $x \in \mathcal{C}$ iff there is a sequence $(a_n)_{n\in\mathbf{N}}$ with $a_n \in \{0,2\}$ for all n, such that $x = \sum_{n\geq 0} \frac{a_n}{3^n}$.

Problem 4. For each of the following tasks, give an example as requested or prove that one does not exist.

(a) An infinite set with two cluster points.
(b) A nonempty open set that is a subset of $\mathbf{Q}$.
(c) A nonempty closed set that is a subset of $\mathbf{Q}$.
(d) Two nonempty disjoint open sets whose union is $\mathbf{R}$.
(e) A set with uncountably many isolated points.
(f) An infinite set with no limit points.
(g) A bounded infinite set with no limit points.
(h) An infinite union of compact sets that is not compact.
(i) An infinite intersection of compact sets that is not compact.

Problem 5. For each of the following sets, determine whether it is open, closed, compact, or none of these. Give a short explanation.

(a) $\{1,2,3\}$;
(b) $\mathbf{Q}$;
(c) $\mathcal{C} \setminus \{0\}$, where $\mathcal{C}$ is the Cantor set;
(d) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$;
(e) $\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots\}$.

Problem 6. Prove that any compact nonempty set contains its sup and inf. Give an example of a set that contains its sup and inf but is not compact.

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Definition.

- The *interior* of a set A, denoted $\operatorname{Int} A$, is the set of points x such that there is a neighborhood of x that is a subset of A.
- The *exterior* of a set A, denoted $\operatorname{Ext} A$, is the set of points x such that there is a neighborhood of x that is a subset of A^c.
- The *boundary* of a set A, denoted ∂A, is the set of points x such that every neighborhood of x contains points from both A and A^c.

Problem 7.

(a) For any set A, $\mathbf{R} = \operatorname{Int} A \cup \partial A \cup \operatorname{Ext} A$, and this is a disjoint union. Prove this.
(b) Find a formula for the interior of a set in terms of the set's closure and its complement's closure. Do the same for the exterior. Prove that your answers are correct.
(c) Find the interior, exterior, and boundary of each of the following sets: $(0,1)$, $[0,1]$, $\mathbf{R}$, $\mathcal{C}$ (the Cantor set).
(d) Prove that $\partial A \cup \operatorname{Int} A = \overline{A}$.
(e) Prove that any open set coincides with its interior and is disjoint from its boundary. Prove that any closed set includes its boundary.

Problem 8. (a) Prove that the only sets with empty boundary are $\mathbf{R}$ and $\varnothing$. (*Hint*: For a point $a \in A$, consider the set $\{x \in \mathbf{R} : (a,x) \subseteq A\}$. Prove that if this set has a supremum, then it belongs to the boundary of A.)
(b) Prove that the only sets that are both open and closed are $\mathbf{R}$ and $\varnothing$.

Other problems.

Problem 9. Say (for this problem only) that a sequence $(a_n)_{n\in\mathbf{N}}$ *u-converges* to L if there is an $N \in \mathbf{N}$ such that for every $\varepsilon > 0$, if $n \geq N$ then $|a_n - L| < \varepsilon$. Show that a sequence u-converges iff it is *eventually constant*, meaning that there is a number $K \in \mathbf{N}$ such that if $m \geq K$ then $a_m = a_K$.

Problem 10. Suppose that $(x_n)_{n\in\mathbf{N}}$ is a bounded sequence. Prove that it has a subsequence converging to $\limsup x_n$.

Problem 11. For each of the following series, determine with proof whether it converges absolutely, conditionally, or not at all.

(a) $\sum \frac{n \sin n}{3^n}$
(b) $\sum \frac{(-1)^n}{\sqrt{n}}$
(c) $\sum \frac{(-1)^n + n}{n^2}$

Problem 12. Prove that if $\sum a_n$ converges and $p > 1$, then $\sum a_n^p$ also converges.