

ON THE SURJECTIVITY OF MOD ℓ REPRESENTATIONS ASSOCIATED TO ELLIPTIC CURVES

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ABSTRACT. Let E be an elliptic curve over the rationals that does not have complex multiplication. For each prime ℓ , the action of the absolute Galois group on the ℓ -torsion points of E can be given in terms of a Galois representation $\rho_{E,\ell}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_\ell)$. An important theorem of Serre says that $\rho_{E,\ell}$ is surjective for all sufficiently large ℓ . In this paper, we describe an algorithm based on Serre's proof that can quickly determine the finite set of primes ℓ for which $\rho_{E,\ell}$ is not surjective. We will also give some improved bounds for Serre's theorem.

1. INTRODUCTION

Let E be a non-CM elliptic curve defined over \mathbb{Q} . For each prime ℓ , let $E[\ell]$ be the ℓ -torsion subgroup of $E(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is a fixed algebraic closure of \mathbb{Q} . The group $E[\ell]$ is a free \mathbb{F}_ℓ -vector space of dimension 2 and there is a natural action of the absolute Galois group $\text{Gal}_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $E[\ell]$ which respects the group structure. After choosing a basis for $E[\ell]$, this action can be expressed in terms of a Galois representation

$$\rho_{E,\ell}: \text{Gal}_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}_\ell).$$

A renowned theorem of Serre shows that $\rho_{E,\ell}$ is surjective for all sufficiently large primes ℓ , cf. [Ser72].

Let $c(E)$ be the smallest integer $n \geq 1$ for which $\rho_{E,\ell}$ is surjective for all primes $\ell > n$. Serre has asked whether the constant $c(E)$ can be bounded independent of E [Ser72, §4.3], and moreover whether $c(E) \leq 37$ always holds [Ser81, p. 399]. We pose a slightly stronger conjecture; first define the set of pairs

$$S_0 := \{ (17, -17^2 \cdot 101^3/2), (17, -17 \cdot 373^3/2^{17}), (37, -7 \cdot 11^3), (37, -7 \cdot 137^3 \cdot 2083^3) \}.$$

Denote by j_E the j -invariant of E/\mathbb{Q} . When $(\ell, j_E) \in S_0$, the curve E has an isogeny of degree ℓ and hence $\rho_{E,\ell}$ is not surjective, cf. [Zyw15] for a description of the image of $\rho_{E,\ell}$.

Conjecture 1.1. *If E is a non-CM elliptic curve over \mathbb{Q} and $\ell > 13$ is a prime satisfying $(\ell, j_E) \notin S_0$, then $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}}) = \text{GL}_2(\mathbb{F}_\ell)$.*

The main goal of this paper is to give a simple and practical algorithm to compute the finite set of primes ℓ for which $\rho_{E,\ell}$ is not surjective. We will focus on the case $\ell > 11$ since using [Zyw15], we can easily compute the group $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$, up to conjugacy in $\text{GL}_2(\mathbb{F}_\ell)$, for all the primes $\ell \leq 11$.

We will also give improved upper bounds for $c(E)$.

Notation. For an elliptic curve E/\mathbb{Q} , denote its j -invariant and conductor by j_E and N_E , respectively. For each prime p for which E has good reduction, define the integer $a_p(E) = |E(\mathbb{F}_p)| - (p+1)$, where $E(\mathbb{F}_p)$ is the \mathbb{F}_p -points of a good model at p . For each good prime $p \neq \ell$, the representation $\rho_{E,\ell}$ is unramified at p and satisfies $\text{tr}(\rho_{E,\ell}(\text{Frob}_p)) \equiv a_p(E) \pmod{\ell}$ and $\det(\rho_{E,\ell}(\text{Frob}_p)) \equiv p \pmod{\ell}$, where $\text{Frob}_p \in \text{Gal}_{\mathbb{Q}}$ is an (arithmetic) Frobenius at p . For primes p for which E has

bad reduction, we set $a_p(E) = 0, 1$ or -1 , if E has additive, split multiplicative or non-split multiplicative reduction, respectively, at p . Let $v_p: \mathbb{Q}_p^\times \rightarrow \mathbb{Z}$ be the valuation for the prime p .

1.1. An algorithm. Fix a non-CM elliptic curve E/\mathbb{Q} . We now explain how to compute a finite set S of primes such that $\rho_{E,\ell}$ is surjective for all primes $\ell \notin S$.

Let $q_1 < \dots < q_d$ be the primes p that satisfy one of the following conditions:

- $p = 2$ and $v_p(j_E)$ is 3, 6 or 9,
- $p \geq 3$ and $v_p(j_E - 1728)$ is positive and odd.

Take any odd prime p for which E has Kodaira symbol I_0 or I_0^* . Equivalently, E/\mathbb{Q} or its quadratic twist by p has good reduction at p ; denote this curve by E_p/\mathbb{Q} .

Let $p_1 < p_2 < p_3 < p_4 < \dots$ be the odd primes such that E has Kodaira symbol I_0 or I_0^* and such that the integer $a_i := |a_{p_i}(E_{p_i})|$ is non-zero. Note that the set of such primes p_i has density 1, cf. [Ser81, Théorème 20].

For integers $i \geq 1$ and $1 \leq j \leq d$, define the following values in \mathbb{F}_2 :

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } q_j \text{ is a square modulo } p_i, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta_i = \begin{cases} 0 & \text{if } -1 \text{ is a square modulo } p_i, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to compute $\alpha_{i,j}$ and β_i ; with respect to the isomorphism $\mathbb{F}_2 \cong \{\pm 1\}$ they are simply Legendre symbols. For each integer $m \geq 1$, let $A_m \in M_{m,d}(\mathbb{F}_2)$ be the $m \times d$ matrix whose (i, j) -th entry is $\alpha_{i,j}$ and let $b_m \in \mathbb{F}_2^m$ be the column vector whose i -th entry is β_i .

Let $r \geq 1$ be the smallest integer for which the linear equation $A_r x = b_r$ has no solution. By Dirichlet's theorem for primes in arithmetic progressions, there is an integer $i_0 \geq 1$ such that $\alpha_{i_0,j} = 0$ for all $1 \leq j \leq d$ and $\beta_{i_0} = 1$. So $r \leq i_0$ and in particular r is well-defined.

Let S be the set of primes ℓ such that $\ell \leq 13$, $(\ell, j_E) \in S_0$, or $a_i \equiv 0 \pmod{\ell}$ for some $1 \leq i \leq r$; it is finite since S_0 is finite and each a_i is non-zero. We will prove the following in §3.

Theorem 1.2. *The representation $\rho_{E,\ell}$ is surjective for all primes $\ell \notin S$.*

We will explain in §6 how to test the surjectivity of $\rho_{E,\ell}$ for the finitely many primes $\ell \in S$.

Example 1.3. We have used Theorem 1.2 to verify Conjecture 1.1 for all elliptic curves E/\mathbb{Q} with conductor at most 360000 (Magma code is given in Appendix A). In fact, for all such curves E/\mathbb{Q} our computations show that $p_r \leq 71$. By the Hasse bound, we have $a_i \leq 2\sqrt{p_i} \leq 2\sqrt{71} < 17$ for $1 \leq i \leq r$. Therefore, the set $S - \{2, 3, 5, 7, 11, 13\}$ is either empty or is $\{\ell\}$ when $(\ell, j_E) \in S_0$. In particular, we did not need to directly check the surjectivity of $\rho_{E,\ell}$ for any exceptional primes $\ell > 13$.

There are earlier results that produce an explicit finite set S that satisfies the conclusion of Theorem 1.2. For example, the bounds of Kraus and Cojocaru mentioned in §1.3 will give such sets S ; however, the resulting sets S can be extremely large and testing surjectivity of $\rho_{E,\ell}$ for the finite number of $\ell \in S$ can be time consuming. Stein verified Conjecture 1.1 for curves of conductor at most 30000 using the bound of Cojocaru, cf. [Ste]; the resulting sets S would typically consist of thousands of primes (this should be contrasted with Example 1.3).

Remark 1.4.

- (i) The set S does not change if we replace E by a quadratic twist and hence it depends only on j_E .

- (ii) In practice, the most time consuming part of computing S is to determine the odd primes p for which $v_p(j_E - 1728)$ is positive and odd; note that the curve E has bad reduction at such primes p . However, observe that we do not need to determine all the primes of bad reduction. (Contrast this with §1.2, where we find an alternate set S when $j_E \notin \mathbb{Z}$ by only using the primes that divide the denominator of j_E .)

1.2. Non-integral j -invariants. Let E/\mathbb{Q} be a non-CM elliptic curve. The following, which will be proved in §4, shows that if $\rho_{E,\ell}$ is not surjective, then the denominator of j_E must be of a special form.

Theorem 1.5. *Let $p_1^{e_1} \cdots p_s^{e_s}$ be the factorization of the denominator of j_E , where the p_i are distinct primes with $e_i > 0$. If $\rho_{E,\ell}$ is not surjective for a prime $\ell > 13$ with $(\ell, j_E) \notin S_0$, then each p_i is congruent to ± 1 modulo ℓ and each e_i is divisible by ℓ .*

Now suppose that the j -invariant of E is not an integer (the theorem is trivial otherwise). Let g be the greatest common divisor of the integers $p_i^2 - 1$ and e_i with $1 \leq i \leq s$. Let S be the set of primes ℓ such that $\ell \leq 13$, $(\ell, j_E) \in S_0$, or $g \equiv 0 \pmod{\ell}$. The set S is finite. The following is a direct consequence of Theorem 1.5.

Proposition 1.6. *If j_E is not an integer, then the representation $\rho_{E,\ell}$ is surjective for all primes $\ell \notin S$.*

Example 1.7. We have verified Conjecture 1.1 for all non-CM elliptic curves E/\mathbb{Q} in the Stein-Watkins database (it consist of 136,924,520 elliptic curves with conductor up to 10^8). Proposition 1.8 sufficed for all E/\mathbb{Q} with $j_E \notin \mathbb{Z}$ (i.e., there were no primes $\ell \in S$ that needed to be checked individually). The integral j -invariants that needed to be considered were handled with the algorithm from §1.1.

We now give some easy bounds for $c(E)$.

Proposition 1.8. *Suppose that j_E is not an integer.*

- (i) *We have $c(E) \leq \max\{17, g\}$.*
- (ii) *We have $c(E) \leq \max\{17, (p+1)/2\}$ for every prime p with $v_p(j_E) < 0$.*
- (iii) *We have $c(E) \leq \max\{17, \log d\}$, where $d \geq 1$ is the denominator of j_E .*

Proof. Note that if $(\ell, j_E) \in S_0$, then $\ell = 17$. The first bound is immediate from Proposition 1.6 since $\max S \leq \max\{17, g\}$. Suppose that p is a prime satisfying $v_p(j_E) < 0$ and $\rho_{E,\ell}$ is not surjective for a prime $\ell > 17$. By Theorem 1.5, we have $p \equiv \pm 1 \pmod{\ell}$. Since $p+1$ and $p-1$ are not primes, we must have $\ell \leq (p+1)/2$. By Theorem 1.5, the denominator d is divisible by p^ℓ and is thus at least $(\ell-1)^\ell$. Hence, $\ell \leq \ell \log(\ell-1) \leq \log d$. \square

Remark 1.9. For any non-CM elliptic curve E/\mathbb{Q} , Masser and Wüstholz [MW93] have shown that $c(E) \leq c(\max\{1, h(j_E)\})^\gamma$, where c and γ are absolute constants (which if computed are very large) and $h(j_E)$ is the logarithmic height of j_E . Proposition 1.8(iii) gives a simple version in the case $j_E \notin \mathbb{Z}$ since $\log d \leq h(j_E)$.

1.3. A bound. We now discuss some bounds for $c(E)$ in terms of the conductor. Kraus [Kra95] proved that

$$c(E) \leq 68 \operatorname{rad}(N_E)(1 + \log \log \operatorname{rad}(N_E))^{1/2}$$

where $\operatorname{rad}(N_E) = \prod_{p|N_E} p$. Using a similar approach, Cojocaru [Coj05] showed that $c(E)$ is at most $\frac{4}{3}\sqrt{6} \cdot N_E \prod_{p|N_E} (1 + 1/p)^{1/2} + 1$. We shall strengthen these bounds with the following theorem which will be proved in §5.

Theorem 1.10. *Let E/\mathbb{Q} be a non-CM elliptic curve that has no primes of multiplicative reduction. Then*

$$c(E) \leq \max \left\{ 37, \frac{2\sqrt{3}}{3} N_E^{1/2} \prod_{p|N_E} \left(\frac{1}{2} + \frac{1}{2p} \right)^{1/2} \right\}.$$

In particular, $c(E) \leq \max \{37, N_E^{1/2}\}$.

Suppose that we are in the excluded case where E/\mathbb{Q} has multiplicative reduction at a prime p . Then the bound $c(E) \leq \max\{37, (p+1)/2\}$ from Proposition 1.8 already gives a sizeable improvement over the bounds of Kraus and Cojocaru.

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2. THE CHARACTER ε_ℓ

Fix a non-CM elliptic curve E/\mathbb{Q} and a prime $\ell > 13$ with $(\ell, j_E) \notin S_0$ such that the representation $\rho_{E,\ell}$ is *not* surjective.

Proposition 2.1 (Serre, Mazur, Bilu-Parent-Rebolledo). *With assumptions as above, the image of $\rho_{E,\ell}$ lies in the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$.*

Before explaining the proposition, let us recall some facts about non-split Cartan subgroups. A non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$ is the image of a homomorphism $\mathbb{F}_{\ell^2}^\times \hookrightarrow \mathrm{Aut}_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^2}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$, where the first map comes from acting by multiplication and the isomorphism arises from some choice of \mathbb{F}_ℓ -basis of \mathbb{F}_{ℓ^2} . Let C be a non-split Cartan subgroup; it is cyclic of order $\ell^2 - 1$ and is uniquely defined up to conjugacy in $\mathrm{GL}_2(\mathbb{F}_\ell)$. Let N be the normalizer of C in $\mathrm{Aut}_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^2}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$; it is the subgroup generated by C and the automorphism $a \mapsto a^\ell$ of \mathbb{F}_{ℓ^2} . In particular, $[N : C] = 2$.

Fix a non-square $\epsilon \in \mathbb{F}_\ell$. After replacing C by a conjugate, one can take C to be the group consisting of matrices of the form $\begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix}$ with $(a, b) \in \mathbb{F}_\ell^2 - \{(0, 0)\}$; the group N is then generated by C and the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For all $g \in N - C$, g^2 is scalar and $\mathrm{tr}(g) = 0$.

Proof of Proposition 2.1. Suppose that $\rho_{E,\ell}$ is not surjective; its image lies in a maximal subgroup H of $\mathrm{GL}_2(\mathbb{F}_\ell)$. We have $\det(\rho_{E,\ell}(\mathrm{Gal}_{\mathbb{Q}})) = \mathbb{F}_\ell^\times$ since the character $\det \circ \rho_{E,\ell}$ corresponds to the Galois action on the ℓ -th roots of unity. Therefore, $\det(H) = \mathbb{F}_\ell^\times$. From [Ser72, §2], we find that, up to conjugation, H is one of the following:

- (a) a Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$,
- (b) the normalizer of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$,
- (c) the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$,
- (d) for $\ell \equiv \pm 3 \pmod{8}$, a subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$ that contains the scalar matrices and whose image in $\mathrm{PGL}_2(\mathbb{F}_\ell)$ is isomorphic to the symmetric group \mathfrak{S}_4 .

That $\rho_{E,\ell}(\mathrm{Gal}_{\mathbb{Q}})$ is not contained in a Borel subgroup when $\ell > 13$ and $(\ell, j_E) \notin S_0$ is a theorem of Mazur, cf. [Maz78]; the modular curves $X_0(17)$ and $X_0(37)$ each have two rational points which are not cusps or CM points and these points are explained by the pairs $(\ell, j_E) \in S_0$. Bilu, Parent and Rebolledo have shown that $\rho_{E,\ell}(\mathrm{Gal}_{\mathbb{Q}})$ cannot be conjugate to a subgroup as in (b), cf. [BPR13]; they make effective the bounds in earlier works of Bilu and Parent using improved isogeny bounds of Gaudron and Rémond. Serre has shown that $\rho_{E,\ell}(\mathrm{Gal}_{\mathbb{Q}})$ cannot be conjugate to a subgroup as in (d), cf. [Ser81, §8.4]. Therefore, the only possibility for H is to be a group as in (c). \square

By Proposition 2.1 and our assumption on $\rho_{E,\ell}$, the image of $\rho_{E,\ell}$ is contained in the normalizer N of a non-split Cartan subgroup C of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Following Serre, we define the quadratic character

$$\varepsilon_\ell: \mathrm{Gal}_{\mathbb{Q}} \xrightarrow{\rho_{E,\ell}} N/C \xrightarrow{\sim} \{\pm 1\}.$$

For each prime p , let I_p be an inertia subgroup of $\mathrm{Gal}_{\mathbb{Q}}$ at p . Recall that ε_ℓ is unramified at p if and only if $\varepsilon_\ell(I_p) = \{1\}$. We now state several basic lemmas concerning the character ε_ℓ . Let q_1, \dots, q_d be the primes from §1.1.

Lemma 2.2.

- (i) *The character ε_ℓ is unramified at ℓ and at all primes $p \notin \{q_1, \dots, q_d\}$.*
- (ii) *If $p \in \{q_1, \dots, q_d\} - \{\ell\}$, then $\rho_{E,\ell}(I_p)$ contains $-I$ and an element of order 4.*

Proof. Take any prime p .

- First suppose that $p = \ell$. Let I'_ℓ be the maximal pro- ℓ subgroup of I_ℓ . We have $\rho_{E,\ell}(I'_\ell) = 1$ since N has cardinality relatively prime to ℓ . The group $\rho_{E,\ell}(I_\ell)$ is cyclic since every finite quotient of the tame inertia group I_ℓ/I'_ℓ is cyclic, see [Ser72, §1.3] for the structure of I_ℓ/I'_ℓ . Fix a generator g of $\rho_{E,\ell}(I_\ell)$. By the proof of [Ser81, p.397 Lemme 18'], the image $\rho_{E,\ell}(I_\ell)$ in $\mathrm{PGL}_2(\mathbb{F}_\ell)$ contains an element of order at least $(\ell - 1)/4 > 2$. The order of the image of g in $\mathrm{PGL}_2(\mathbb{F}_\ell)$ is greater than 2, so g^2 is not a scalar matrix. However, g^2 is a scalar matrix for all $g \in N - C$. So g belongs to C and thus $\rho_{E,\ell}(I_\ell) \subseteq C$. Therefore, ε_ℓ is unramified at ℓ .
- Suppose that $p \neq \ell$ and that E has good reduction at p . We have $\rho_{E,\ell}(I_p) = \{I\} \subseteq C$ since $\rho_{E,\ell}$ is unramified at such primes p . Therefore, ε_ℓ is unramified at p .
- Suppose that $p \neq \ell$ and that $v_p(j_E) < 0$. Using a Tate curve, we shall show in §4 that ε_ℓ is unramified at p (and moreover that $\varepsilon_\ell(\mathrm{Frob}_p) \equiv p \pmod{\ell}$); the proof will use the definition of ε_ℓ but none of the successive lemmas in this section.
- Finally suppose that $p \neq \ell$ is a prime for which E has bad reduction at p and $v_p(j_E) \geq 0$. Choose a minimal Weierstrass model of E/\mathbb{Q} and let Δ , c_4 and c_6 be the standard invariants attached to this model as given in [Sil09, III §1].

Let Φ_p be the image of I_p under $\rho_{E,\ell}$. We can identify Φ_p with $\mathrm{Gal}(L/\mathbb{Q}_p^{\mathrm{un}})$ where L is the smallest extension of $\mathbb{Q}_p^{\mathrm{un}}$ for which E base extended to L has good reduction. Moreover, one knows that Φ_p is isomorphic to a subgroup of $\mathrm{Aut}(\tilde{E})$ where $\tilde{E}/\overline{\mathbb{F}}_p$ is the reduction of E/L , cf. [Ser72, §5.6]. We have $\Phi_p \subseteq \mathrm{SL}_2(\mathbb{F}_\ell)$ since $\det \circ \rho_{E,\ell}$ is ramified only at the prime ℓ . In particular, if there is an element in Φ_p with order 2, then it is $-I$.

Consider $p \geq 5$. The group $\mathrm{Aut}(\tilde{E})$ is cyclic of order 2, 4 or 6, so Φ_p is cyclic of order 2, 3, 4 or 6. We have $j_E - 1728 = c_6^2/\Delta$, so $v_p(j_E - 1728) \equiv v_p(\Delta) \pmod{2}$. From [Ser72, §5.6], we find that Φ_p has order 2, 3 or 6 if and only if $v_p(j - 1728)$ is even.

Consider $p = 3$. The group $\mathrm{Aut}(\tilde{E})$ is now either cyclic of order 2, 4 or 6, or is a non-abelian group of order 12 (it is a semi-direct product of a cyclic group of order 4 by a distinguished subgroup of order 3). Using that $v_p(j_E - 1728) \equiv v_p(\Delta) \pmod{2}$ and Théorème 1 of [Kra90], we find that Φ_p has order 2, 3 or 6 if and only if $v_p(j - 1728)$ is even.

Consider $p = 2$. Then the group $\mathrm{Aut}(\tilde{E})$, and hence also Φ_p is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$. The group Φ_p is either cyclic of order 2, 3, 4 or 6, isomorphic to the order 8 group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$, or is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. We have $j_E = c_4^3/\Delta$ and hence $v_2(j_E) = 3v_2(c_4) - v_2(\Delta)$. Checking all the cases in the corollary to Théorème 3 of [Kra90], we find Φ_p has order 2, 3, 6 or 24 if and only if $v_2(j_E) \notin \{3, 6, 9\}$. The group $\mathrm{SL}_2(\mathbb{F}_3)$ is not isomorphic to a subgroup of N since $\mathrm{SL}_2(\mathbb{F}_3)$ is non-abelian and has no index 2 normal subgroups. Since $\Phi_p \subseteq N$, this proves that $|\Phi_p| \neq 24$.

Now suppose that $p \notin \{q_1, \dots, q_d\}$. From the above computations and our choice of q_j , we find that Φ_p has order 2, 3 or 6. If Φ_p has order 2 or 6, then $-I \in \Phi_p$. Since $-I \in C$ and $[N : C] = 2$,

we deduce that Φ_p is a subgroup of C . Therefore, ε_ℓ is unramified at p . This completes the proof of (i).

Finally suppose that $p \in \{q_1, \dots, q_d\}$ (and $p \neq \ell$). Then Φ_p is cyclic of order 4, or has order 12 ($p = 3$), or has order 8 ($p = 2$). In all these cases, Φ_p contains an element g of order 4. The element g^2 of order 2 in C must be $-I$. This completes the proof of (ii). \square

Remark 2.3. If $\ell \equiv 1 \pmod{4}$, then we claim that ε_ℓ is ramified at a prime p if and only if $p \in \{q_1, \dots, q_d\} - \{\ell\}$. One direction of the claim is immediate from Lemma 2.2(i). Now take any prime $p \in \{q_1, \dots, q_r\} - \{\ell\}$. Suppose that ε_ℓ is unramified at p and hence $\Phi_p := \rho_{E,\ell}(I_p)$ is a subgroup of C . We have $\Phi_p \subseteq C \cap \mathrm{SL}_2(\mathbb{F}_\ell)$ since $\det \circ \rho_{E,\ell}$ is ramified only at ℓ . The group $C \cap \mathrm{SL}_2(\mathbb{F}_\ell)$ has no elements of order 4 since it is cyclic of order $\ell + 1$ and $\ell + 1 \equiv 2 \pmod{4}$. This contradicts Lemma 2.2(ii), so ε_ℓ is indeed ramified at p .

Lemma 2.4. *There are unique integers $e_1, \dots, e_d \in \{0, 1\}$ such that $\varepsilon_\ell(\mathrm{Frob}_p) = \left(\frac{-1}{p}\right) \cdot \prod_{j=1}^d \left(\frac{q_j}{p}\right)^{e_j}$ for all odd primes $p \nmid q_1 \cdots q_d$. In particular, $\varepsilon_\ell \neq 1$.*

Proof. There is a unique squarefree integer D such that $\varepsilon_\ell(\mathrm{Frob}_p) = \left(\frac{-D}{p}\right)$ for all odd primes $p \nmid D$. Let q be any prime dividing D . The character ε_ℓ is ramified at q , so $q = q_j$ for some j by Lemma 2.2. Therefore, D divides $q_1 \cdots q_d$.

It remains to show that D is positive. It suffices to show that $\varepsilon_\ell(c) = -1$, where $c \in \mathrm{Gal}_{\mathbb{Q}}$ corresponds to complex conjugation under a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Set $g := \rho_{E,\ell}(c)$. We have $g^2 = I$ since c has order 2. The matrix g has determinant -1 since the character $\det \circ \rho_{E,\ell}$ corresponds to the Galois action on the ℓ -th roots of unity. The Cartan subgroup C is cyclic since it is non-split, so the only elements of C with order 1 or 2 are I and $-I$. Since $\det(\pm I) = 1$, we deduce that $g \notin C$ and hence $\varepsilon_\ell(c) = -1$ as claimed. \square

Lemma 2.5. *Let p be a prime for which E has good reduction. If $a_p(E) \not\equiv 0 \pmod{\ell}$, then $\varepsilon_\ell(\mathrm{Frob}_p) = 1$.*

Proof. That $a_p(E) \equiv 0 \pmod{\ell}$ for every good prime p satisfying $\varepsilon(\mathrm{Frob}_p) = -1$ is [Ser72, p.317(c5)]; for $p \neq \ell$, this follows by noting that $\mathrm{tr}(g) = 0$ for all $g \in N - C$. \square

3. PROOF OF THEOREM 1.2

Replacing E/\mathbb{Q} by a quadratic twist does not change the set S or the set of primes ℓ for which $\rho_{E,\ell}$ is not surjective. We may thus assume that E has no odd primes p with Kodaira type I_0^* . So for each p_i , we have $a_i = |a_{p_i}(E)|$.

Suppose that $\ell \notin S$ is a prime for which $\rho_{E,\ell}$ is not surjective. From our choice of ℓ , Proposition 2.1 implies that the image of $\rho_{E,\ell}$ is contained in the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Let $\varepsilon_\ell: \mathrm{Gal}_{\mathbb{Q}} \rightarrow \{\pm 1\}$ be the corresponding quadratic character. By Lemma 2.4, there are unique $e_1, \dots, e_d \in \{0, 1\}$ such that $\varepsilon_\ell(\mathrm{Frob}_p) = \left(\frac{-1}{p}\right) \cdot \prod_{j=1}^d \left(\frac{q_j}{p}\right)^{e_j}$ for all primes $p \nmid 2q_1 \cdots q_d$.

Now consider $p = p_i$ with $1 \leq i \leq r$. We have $|a_{p_i}(E)| = a_i \not\equiv 0 \pmod{\ell}$ since $\ell \notin S$. Lemma 2.5 implies that $\varepsilon_\ell(\mathrm{Frob}_{p_i}) = 1$ for all $1 \leq i \leq r$. Therefore,

$$\prod_{j=1}^d \left(\frac{q_j}{p_i}\right)^{e_j} = \left(\frac{-1}{p_i}\right)$$

for all $1 \leq i \leq r$. Using the isomorphism $\{\pm 1\} \cong \mathbb{F}_2$, this is equivalent to having $\sum_{j=1}^d \alpha_{i,j} e_j = \beta_i$ for all $1 \leq i \leq r$. This shows that the equation $A_r x = b_r$ has a solution in \mathbb{F}_2^d . This is a contradiction since the equation $A_r x = b_r$ has no solution by our choice of r . Therefore, the representation $\rho_{E,\ell}$ must be surjective for all $\ell \notin S$.

4. PROOF OF THEOREM 1.5

Take any prime p that divides the denominator of j_E . Everything that follows is a local argument, so by base extending we shall view E as an elliptic curve over \mathbb{Q}_p ; we have a Galois representation $\rho_{E,\ell}: \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(\mathbb{F}_\ell)$. There exists an element $q \in \mathbb{Q}_p$ with $v_p(q) = -v_p(j_E) > 0$ such that

$$j_E = (1 + 240 \sum_{n \geq 1} n^3 q^n / (1 - q^n))^3 / (q \prod_{n \geq 1} (1 - q^n)^{24}) = q^{-1} + 744 + 196884q + \dots;$$

let \mathcal{E}/\mathbb{Q}_p be the Tate curve associated to q , cf. [Sil94, V§3]. It is an elliptic curve with j -invariant j_E and the group $\mathcal{E}(\overline{\mathbb{Q}_p})$ is isomorphic to $\overline{\mathbb{Q}_p}^\times / \langle q \rangle$ as a $\text{Gal}_{\mathbb{Q}_p}$ -module. In particular, the ℓ -torsion subgroup $\mathcal{E}[\ell]$ is isomorphic as an $\mathbb{F}_\ell[\text{Gal}_{\mathbb{Q}_p}]$ -module to the subgroup of $\overline{\mathbb{Q}_p}^\times / \langle q \rangle$ generated by an ℓ -th root of unity ζ_ℓ and a chosen ℓ -th root $q^{1/\ell}$ of q . Let $\alpha: \text{Gal}_{\mathbb{Q}_p} \rightarrow \mathbb{F}_\ell^\times$ and $\beta: \text{Gal}_{\mathbb{Q}_p} \rightarrow \mathbb{F}_\ell$ be the maps defined so that

$$\sigma(\zeta_\ell) = \zeta_\ell^{\alpha(\sigma)} \quad \text{and} \quad \sigma(q^{1/\ell}) = \zeta_\ell^{\beta(\sigma)} q^{1/\ell}$$

for all $\sigma \in \text{Gal}_{\mathbb{Q}_p}$. So for an appropriate choice of basis for $\mathcal{E}[\ell]$, we have $\rho_{\mathcal{E},\ell}(\sigma) = \begin{pmatrix} \alpha(\sigma) & \beta(\sigma) \\ 0 & 1 \end{pmatrix}$ for $\sigma \in \text{Gal}_{\mathbb{Q}_p}$. The curves E and \mathcal{E} are quadratic twists of each other over \mathbb{Q}_p since they are non-CM curves with the same j -invariant. So there is a character $\chi: \text{Gal}_{\mathbb{Q}_p} \rightarrow \{\pm 1\}$ such that, after an appropriate choice of basis for $E[\ell]$, we have

$$\rho_{E,\ell}(\sigma) = \chi(\sigma) \begin{pmatrix} \alpha(\sigma) & \beta(\sigma) \\ 0 & 1 \end{pmatrix}$$

for all $\sigma \in \text{Gal}_{\mathbb{Q}_p}$.

Now assume that $\rho_{E,\ell}$ is not surjective for a prime $\ell > 13$ with $(\ell, j_E) \notin S_0$. By Proposition 2.1, the image of $\rho_{E,\ell}$ is contained in the normalizer N of a non-split Cartan subgroup C of $\text{GL}_2(\mathbb{F}_\ell)$. Let $\varepsilon_\ell: \text{Gal}_{\mathbb{Q}} \rightarrow \{\pm 1\}$ be the corresponding quadratic character.

Since C is non-split, the only matrices in C with eigenvalue 1 or -1 are $\pm I$. So if $\rho_{E,\ell}(\sigma)$ belongs to C , then $\alpha(\sigma) = 1$ and $\beta(\sigma) = 0$. If $\rho_{E,\ell}(\sigma)$ belongs to $N - C$, then $\alpha(\sigma) = -1$ since every matrix in $N - C$ has trace 0. This proves that α takes values in $\{\pm 1\}$ and that $\alpha(\sigma) \equiv \varepsilon_\ell(\sigma) \pmod{\ell}$ for all $\sigma \in \text{Gal}_{\mathbb{Q}_p}$. If $\ell = p$, then $\alpha(\text{Gal}_{\mathbb{Q}_p}) = \mathbb{F}_\ell^\times$ which is impossible since $\ell > 13$ and α takes values in $\{\pm 1\}$. So $\ell \neq p$ and hence $\alpha(\text{Frob}_p) \equiv p \pmod{\ell}$. Therefore, ε_ℓ is unramified at p and $\varepsilon_\ell(\text{Frob}_p) \equiv \alpha(\text{Frob}_p) \equiv p \pmod{\ell}$. In particular, we must have $p \equiv \pm 1 \pmod{\ell}$.

It remains to prove that $e := -v_p(j_E)$ is divisible by ℓ . The matrices I and $-I$ are the only elements of N that have eigenvalue 1 or -1 with multiplicity 2. Since $\alpha(\text{Gal}_{\mathbb{Q}_p(\zeta_\ell)}) = 1$, we must have $\beta(\text{Gal}_{\mathbb{Q}_p(\zeta_\ell)}) = 0$ and hence $q^{1/\ell} \in \mathbb{Q}_p(\zeta_\ell)$. Extend the valuation v_p of \mathbb{Q}_p to $\mathbb{Q}_p(\zeta_\ell)$. Since $\mathbb{Q}_p(\zeta_\ell)/\mathbb{Q}_p$ is an unramified extension (we saw above that $p \neq \ell$), we deduce that $v_p(q^{1/\ell})$ belongs to \mathbb{Z} and hence $e = -v_p(j_E) = v_p(q) = \ell v_p(q^{1/\ell}) \in \ell\mathbb{Z}$.

5. PROOF OF THEOREM 1.10

Suppose that $\rho_{E,\ell}$ is not surjective for a prime $\ell > 13$ with $(\ell, j_E) \notin S_0$. We can then define a quadratic character $\varepsilon_\ell: \text{Gal}_{\mathbb{Q}} \rightarrow \{\pm 1\}$ as in §2. Let E'/\mathbb{Q} be the elliptic curve obtained by twisting E/\mathbb{Q} by ε_ℓ .

Lemma 5.1. *The elliptic curves E and E' have the same conductors.*

Proof. Take any prime p . Lemma 1 of [Kra95] says that E and E' have the same reduction type (i.e., good, additive or multiplicative) at p . This proves that $\text{ord}_p(N_E) = \text{ord}_p(N_{E'})$ for $p \geq 5$. To prove this equality for $p = 2$ and 3 , we need to check that the wild part of the conductors of E and E' at p agree; for a description of the wild part of the conductor at p , see [Sil94, IV§10].

For our prime $p \leq 3$, it suffices to show that the groups $\rho_{E,\ell}(I_p)$ and $\rho_{E',\ell}(I_p)$ are conjugate in $\mathrm{GL}_2(\mathbb{F}_\ell)$. After choosing appropriate bases of $E[\ell]$ and $E'[\ell]$, we may assume that $\rho_{E',\ell} = \varepsilon_\ell \cdot \rho_{E,\ell}$. If ε_ℓ is unramified at p , then $\rho_{E',\ell}(I_p) = \rho_{E,\ell}(I_p)$. We always have $\pm \rho_{E',\ell}(I_p) = \pm \rho_{E,\ell}(I_p)$. So if ε_ℓ is ramified at p , then Lemma 2.2(ii) implies that $\rho_{E',\ell}(I_p) = \pm \rho_{E,\ell}(I_p) = \rho_{E,\ell}(I_p)$. \square

By Lemma 5.1, the elliptic curves E and E' have the same conductor; denote it by N . By the modularity theorem (proved by Wiles, Taylor, Breuil, Conrad and Diamond), there are newforms f and $g \in S_2(\Gamma_0(N))$ corresponding to E and E' , respectively. Let $a_n(f)$ and $a_n(g)$ be the n -th Fourier coefficient of f and g at the cusp $i\infty$. The following lemma gives a Sturm bound for a prime q that satisfies $a_q(f) \neq a_q(g)$. Note that f and g are distinct since $\varepsilon \neq 1$ (by Lemma 2.4) and since E and E' are non-CM.

Lemma 5.2. *Let f and g be distinct normalized newforms in $S_2(\Gamma_0(N))$. Then there exists a prime q such that*

$$(5.1) \quad q \leq \frac{N}{3} \prod_{p|N} \left(\frac{1}{2} + \frac{1}{2p} \right) - 1$$

and $a_q(f) \neq a_q(g)$.

Proof. Consider the modular curve $X_0(N)$ defined over \mathbb{C} . Its complex points form a Riemann surface obtained by quotienting the complex upper-half plane by $\Gamma_0(N)$ and then compactifying by adding cusps. For each prime power $q = p^e$ such that $p^e \parallel N$, let W_q be a matrix of the form $\begin{pmatrix} qa & b \\ Nc & qd \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ that has determinant q . The matrix W_q normalizes $\Gamma_0(N)$ and thus induces an automorphism of $X_0(N)$. Let $W(N)$ be the subgroup of $\mathrm{Aut}(X_0(N))$ generated by the $\{W_{p^e}\}_{p^e \parallel N}$. The group $W(N)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ where r is the number of distinct prime factors of N [AL70, Lemma 9]. The group $W(N)$ permutes the cusps of $X_0(N)$ and the stabilizer of the cusp $i\infty$ is trivial.

For the newform f , consider the holomorphic differential form $\eta = f(z)dz$ on $X_0(N)$. For each automorphism $w \in W(N)$, there is a $\lambda_w(f) \in \{\pm 1\}$ such that $\eta(wz) = \lambda_w(f)\eta(z)$, cf. [AL70, Theorem 3]. Similarly, we have values $\lambda_w(g) \in \{\pm 1\}$ for $w \in W(N)$.

Let H be the set of $w \in W(N)$ for which $\lambda_w(f) = \lambda_w(g)$; it is a subgroup of $W(N)$ of cardinality 2^r or 2^{r-1} . The holomorphic differential form $\omega := (f(z) - g(z))dz$ is non-zero since f and g are distinct. Let $K = \mathrm{div}(\omega)$ be the corresponding (effective) divisor on $X_0(N)$; it has degree $2g_{X_0(N)} - 2$ where $g_{X_0(N)}$ is the genus of $X_0(N)$. Therefore,

$$\sum_P \mathrm{ord}_P(\omega) \leq 2g_{X_0(N)} - 2$$

where the sum is over the cusps of $X_0(N)$. For a fixed automorphism $w \in H$, we have a cusp $P = w \cdot i\infty$. From our choice of H , we find that $\omega(wz) = \pm \omega(z)$ and thus $\mathrm{ord}_P(\omega) = \mathrm{ord}_{i\infty}(\omega)$. Therefore,

$$2^{r-1} \mathrm{ord}_{i\infty}(\omega) \leq |H| \mathrm{ord}_{i\infty}(\omega) \leq 2g_{X_0(N)} - 2 \leq \frac{N}{6} \prod_{p|N} (1 + 1/p) - 2^r$$

where the last inequality uses an explicit formula for $g_{X_0(N)}$ [Shi94, Prop. 1.40] and that $X_0(N)$ has at least 2^r cusps. Let n be the smallest positive integer for which the Fourier coefficients $a_n(f)$ and $a_n(g)$ disagree. We have $\mathrm{ord}_{i\infty}(\omega) = n - 1$, and hence

$$n \leq \frac{1}{2^r} \frac{N}{3} \prod_{p|N} (1 + 1/p) - 1.$$

If n is prime, then we are done. If n is composite with $a_n(f) \neq a_n(g)$, then $a_q(f) \neq a_q(g)$ for some prime q dividing n (since f and g are normalized eigenforms, we know that their Fourier coefficients are multiplicative and are defined recursively for prime powers indices). \square

Remark 5.3. If f and g are distinct modular forms on $\Gamma_0(N)$ of weight 2, then the same proof, but only looking at the cusp $i\infty$, shows that there is an integer $n \leq \frac{N}{6} \prod_{p|N} (1 + \frac{1}{p})$ such that $a_n(f) \neq a_n(g)$. This is the bound used in [Coj05] and [Kra95]; though possibly working with a larger N .

By Lemma 5.2, there is a prime q satisfying (5.1) such that $a_q(E) = a_q(f) \neq a_q(g) = a_q(E')$. Since $a_p(E) = a_p(E') = 0$ for primes of additive reduction, we find that E has either good or multiplicative reduction at q . By assumption, E has no primes of multiplicative reduction, so E has good reduction at q .

Since $a_q(E) \neq a_q(E') = \varepsilon_\ell(\text{Frob}_q)a_q(E)$, we deduce that $\varepsilon_\ell(\text{Frob}_q) = -1$ and $a_q(E) \neq 0$. By Lemma 2.5, we find that $a_q(E) \equiv 0 \pmod{\ell}$. The Hasse bound then implies that

$$\ell \leq |a_q(E)| \leq 2\sqrt{q} \leq 2 \sqrt{\frac{N}{3} \prod_{p|N} \left(\frac{1}{2} + \frac{1}{2p}\right)} = \frac{2\sqrt{3}}{3} N^{1/2} \prod_{p|N} \left(\frac{1}{2} + \frac{1}{2p}\right)^{1/2}.$$

Since N is divisible by some prime (there is no elliptic curve over \mathbb{Q} with good reduction everywhere), we have $\ell \leq \frac{2\sqrt{3}}{3} N^{1/2} \left(\frac{1}{2} + \frac{1}{4}\right)^{1/2} = N^{1/2}$.

6. REMAINING PRIMES

Fix a non-CM elliptic curve E/\mathbb{Q} . In this section, we explain how to determine whether $\rho_{E,\ell}$ is surjective for a fixed prime ℓ . Combined with Theorem 1.2 (or possibly Proposition 1.6), this gives a method to compute the (finite) set of primes ℓ for which $\rho_{E,\ell}$ is not surjective.

We will also discuss the surjectivity of the ℓ -adic representations of E in §6.4.

6.1. Primes $\ell \leq 11$. Let \mathcal{E} be the elliptic curve over \mathbb{Q} defined by the Weierstrass equation $y^2 + y = x^3 - x^2 - 7x + 10$ and let \mathcal{O} be the point at infinity. The Mordell-Weil group $\mathcal{E}(\mathbb{Q})$ is an infinite cyclic group generated by the point $(4, 5)$. Let $J: \mathcal{E} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \cup \{\infty\}$ be the morphism given by

$$J(x, y) = (f_1 f_2 f_3 f_4)^3 / (f_5^2 f_6^{11}),$$

where

$$\begin{aligned} f_1 &= x^2 + 3x - 6, & f_2 &= 11(x^2 - 5)y + (2x^4 + 23x^3 - 72x^2 - 28x + 127), \\ f_3 &= 6y + 11x - 19, & f_4 &= 22(x - 2)y + (5x^3 + 17x^2 - 112x + 120), \\ f_5 &= 11y + (2x^2 + 17x - 34), & f_6 &= (x - 4)y - (5x - 9). \end{aligned}$$

For $\ell \leq 11$, the following gives a criterion to determine whether $\rho_{E,\ell}$ is surjective or not.

Proposition 6.1. *Let E/\mathbb{Q} be a non-CM elliptic curve.*

- (i) *The representation $\rho_{E,2}$ is not surjective if and only if $j_E = 256(t + 1)^3/t$ or $j_E = t^2 + 1728$ for some $t \in \mathbb{Q}$.*
- (ii) *The representation $\rho_{E,3}$ is not surjective if and only if $j_E = 27(t + 1)(t + 9)^3/t^3$ or $j_E = t^3$ for some $t \in \mathbb{Q}$.*
- (iii) *The representation $\rho_{E,5}$ is not surjective if and only if*

$$j_E = \frac{5^3(t + 1)(2t + 1)^3(2t^2 - 3t + 3)^3}{(t^2 + t - 1)^5}, \quad j_E = \frac{5^2(t^2 + 10t + 5)^3}{t^5} \quad \text{or} \quad j_E = t^3(t^2 + 5t + 40)$$

for some $t \in \mathbb{Q}$.

(iv) The representation $\rho_{E,7}$ is not surjective if and only if

$$j_E = \frac{t(t+1)^3(t^2-5t+1)^3(t^2-5t+8)^3(t^4-5t^3+8t^2-7t+7)^3}{(t^3-4t^2+3t+1)^7},$$

$$j_E = \frac{64t^3(t^2+7)^3(t^2-7t+14)^3(5t^2-14t-7)^3}{(t^3-7t^2+7t+7)^7} \quad \text{or}$$

$$j_E = \frac{(t^2+245t+2401)^3(t^2+13t+49)}{t^7}$$

for some $t \in \mathbb{Q}$.

(v) The representation $\rho_{E,11}$ is not surjective if and only if $j_E \in \{-11^2, -11 \cdot 131^3\}$ or $j_E = J(P)$ for some $P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}$.

(vi) If j_E is an integer, then $\rho_{E,11}$ is not surjective if and only if $j_E \in \{-11^2, -11 \cdot 131^3\}$. If j_E is not an integer and $\rho_{E,11}$ is not surjective, then the denominator of j_E is of the form $p_1^{e_1} \cdots p_s^{e_s}$ with p_i distinct primes such that $p_i \equiv \pm 1 \pmod{11}$ and $e_i \equiv 0 \pmod{11}$.

Proof. Parts (i)–(v) are consequence of the theorems from [Zyw15]; one need only consider the maximal subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Note that the normalizer of a split Cartan subgroup in $\mathrm{GL}_2(\mathbb{F}_3)$ is not a maximal subgroup. The normalizer of a split Cartan subgroup in $\mathrm{GL}_2(\mathbb{F}_5)$ lies in a maximal subgroup of $\mathrm{GL}_2(\mathbb{F}_5)$ whose image in $\mathrm{PGL}_2(\mathbb{F}_5)$ is isomorphic to \mathfrak{S}_4 .

The curve \mathcal{E} and the map J come from Halberstadt's description of $X_{\mathrm{ns}}^+(11)$ in [Hal98]. In particular, the group $\rho_{E,11}(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_{11})$ if and only if $j_E = J(P)$ for some $P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}$. In [ST12], it is shown that if $J(P)$ is an integer with $P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}$, then $J(P)$ is the j -invariant of a CM elliptic curve; this proves the first part of (vi). For the second part of (vi), note that the proof of Theorem 1.5 applies verbatim. \square

Remark 6.2. In [Zyw15], we give explicit polynomials $A, B, C \in \mathbb{Q}[X]$ of degree 55 such that $j_E = J(P)$ for some point $P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}$ if and only if the polynomial $A(X)j_E^2 + B(X)j_E + C(X) \in \mathbb{Q}[X]$ has a root. So it straightforward to determine whether $j_E = J(P)$ for some $P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}$.

6.2. The prime $\ell = 13$.

Proposition 6.3.

(i) The representation $\rho_{E,13}$ is not surjective if

$$j_E = 2^4 \cdot 5 \cdot 13^4 \cdot 17^3 / 3^{13},$$

$$j_E = -2^{12} \cdot 5^3 \cdot 11 \cdot 13^4 / 3^{13},$$

$$j_E = 2^{18} \cdot 3^3 \cdot 13^4 \cdot 127^3 \cdot 139^3 \cdot 157^3 \cdot 283^3 \cdot 929 / (5^{13} \cdot 61^{13}), \quad \text{or}$$

$$j_E = (t^2 + 5t + 13)(t^4 + 7t^3 + 20t^2 + 19t + 1)^3 / t \quad \text{for some } t \in \mathbb{Q}.$$

(ii) The representation $\rho_{E,13}$ is surjective if and only if all the following conditions hold:

- there is a prime $p \nmid 13N_E$ such that $a_p(E) \not\equiv 0 \pmod{13}$ and such that $a_p(E)^2 - 4p$ is a non-zero square modulo 13,
- there is a prime $p \nmid 13N_E$ such that $a_p(E) \not\equiv 0 \pmod{13}$ and such that $a_p(E)^2 - 4p$ is a non-square modulo 13,
- there is prime $p \nmid 13N_E$ such that the image of $a_p(E)^2/p$ in \mathbb{F}_{13} is not 0, 1, 2 and 4, and is not a root of $x^2 - 3x + 1$.

Proof. Part (i) is explained in [Zyw15]; the first three exceptional j -invariants come from [BC14]. Part (ii) is a direct consequence of Proposition 19 of [Ser72] and the Chebotarev density theorem. \square

Consider a non-CM elliptic curve E/\mathbb{Q} . Suppose that j_E is not one of those given in Proposition 6.3(i); if it was then $\rho_{E,13}$ would not be surjective. Conjecturally, the representation $\rho_{E,13}$ will be surjective and hence this should be checkable using the criterion of Proposition 6.3(ii).

If the surjectivity is unknown even after computing $a_p(E)$ for many primes $p \nmid 13N_E$, then one can do a direct computation. The representation $\rho_{E,13}$ is surjective if and only if the image of $\rho_{E,13}(\text{Gal}_{\mathbb{Q}})$ in $\text{GL}_2(\mathbb{F}_{13})/\{\pm I\}$ is the full group $\text{GL}_2(\mathbb{F}_{13})/\{\pm I\}$. For a given Weierstrass equation $y^2 = x^3 + ax + b$ for E/\mathbb{Q} one can compute the division polynomial of E at the prime 13; it is the monic polynomial $f(X) \in \mathbb{Q}[X]$ whose roots are the x -coordinates of the elements of order 13 in $E(\overline{\mathbb{Q}})$. The Galois group of $f(x)$ is isomorphic to the image of $\rho_{E,13}(\text{Gal}_{\mathbb{Q}})$ in $\text{GL}_2(\mathbb{F}_{13})/\{\pm I\}$ and be computed directly. (For example, this was how the author found the interesting j -invariants $2^4 \cdot 5 \cdot 13^4 \cdot 17^3/3^{13}$ and $-2^{12} \cdot 5^3 \cdot 11 \cdot 13^4/3^{13}$ before [BC14] was available.)

Alternatively, if $\rho_{E,13}$ was not surjective, then one could construct a new rational point on one of the explicit genus 3 curves in [BC14] or [Bar14].

6.3. A surjectivity criterion for primes $\ell > 13$. Fix a prime $\ell > 13$.

Proposition 6.4. *The representation $\rho_{E,\ell}$ is surjective if and only if $(\ell, j_E) \notin S_0$ and there is a prime $p \nmid N_E \ell$ such that $a_p(E) \not\equiv 0 \pmod{\ell}$ and $a_p(E)^2 - 4p$ is a non-zero square modulo ℓ .*

Proof. As noted in the introduction, the representation $\rho_{E,\ell}$ is not surjective when $(\ell, j_E) \in S_0$. So assume that $(\ell, j_E) \notin S_0$. First suppose that there is a prime $p \nmid N_E \ell$ such that $a_p(E) \not\equiv 0 \pmod{\ell}$ and $a_p(E)^2 - 4p$ is a non-zero square modulo ℓ . With $g := \rho_{E,\ell}(\text{Frob}_p)$, we have $\text{tr}(g) \neq 0$ and $\text{tr}(g)^2 - 4 \det(g)$ a non-zero square. Let N be the normalizer of a non-split Cartan subgroup C of $\text{GL}_2(\mathbb{F}_{\ell})$. For all $A \in N - C$, we have $\text{tr}(A) = 0$. For all $A \in C$, the value $\text{tr}(A)^2 - 4 \det(A) \in \mathbb{F}_{\ell}$ is either zero or a non-square. So $g \notin N$, and hence $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ is not a subgroup of the normalizer of a non-split Cartan. Therefore, $\rho_{E,\ell}$ is surjective by Proposition 2.1.

Now suppose that $\rho_{E,\ell}$ is surjective. There are matrices $A \in \text{GL}_2(\mathbb{F}_{\ell})$ so that $\text{tr}(A) \neq 0$ and $\text{tr}(A)^2 - 4 \det(A)$ is a non-zero square. That primes p as in the statement of the proposition occur is then a consequence of the Chebotarev density theorem. \square

Assuming that Conjecture 1.1 holds, the criterion of Proposition 6.4 will always apply for some p and prove that $\rho_{E,\ell}$ is surjective when $(\ell, j_E) \notin S_0$. Using an explicit version of the Chebotarev density theorem, one could even give a bound for p .

However, if $(\ell, j_E) \notin S_0$ and the surjectivity is unknown even after computing $a_p(E)$ for many primes $p \nmid N_E \ell$, then one can do a direct computation. The representation $\rho_{E,\ell}$ is surjective if and only if the image of $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ in $\text{GL}_2(\mathbb{F}_{\ell})/\{\pm I\}$ is the full group $\text{GL}_2(\mathbb{F}_{\ell})/\{\pm I\}$. For a given Weierstrass equation $y^2 = x^3 + ax + b$ for E/\mathbb{Q} one can compute the division polynomial of E at the prime ℓ ; it is the monic polynomial $f(X) \in \mathbb{Q}[X]$ whose roots are the x -coordinates of the elements of order ℓ in $E(\overline{\mathbb{Q}})$. The Galois group of $f(x)$ is isomorphic to the image of $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ in $\text{GL}_2(\mathbb{F}_{\ell})/\{\pm I\}$ and be computed directly.

6.4. ℓ -adic surjectivity. For each integer $n \geq 1$, let $E[\ell^n]$ be the group of ℓ^n -torsion in $E(\overline{\mathbb{Q}})$. The Tate module $T_{\ell}(E)$ of E is the inverse limit of the groups $E[\ell^n]$ with respect to the transition maps $E[\ell^{n+1}] \rightarrow E[\ell^n]$, $P \mapsto \ell P$. The Tate module $T_{\ell}(E)$ is a free \mathbb{Z}_{ℓ} -module of rank 2 with a natural $\text{Gal}_{\mathbb{Q}}$ -action. Let $\rho_{E,\ell^{\infty}}: \text{Gal}_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathbb{Z}_{\ell}}(T_{\ell}(E)) \cong \text{GL}_2(\mathbb{Z}_{\ell})$ be the representation describing this Galois action. Using the results of this paper, and the following lemma, it is straightforward to compute the (finite) set of primes ℓ for which $\rho_{E,\ell^{\infty}}$ is not surjective.

Lemma 6.5. *Let E/\mathbb{Q} be a non-CM elliptic curve.*

- (i) The representation $\rho_{E,2^\infty}$ is not surjective if and only if $\rho_{E,2}$ is not surjective or j_E is of the form

$$-4t^3(t+8), \quad -t^2+1728, \quad 2t^2+1728 \quad \text{or} \quad -2t^2+1728$$

for some $t \in \mathbb{Q}$.

- (ii) The representation $\rho_{E,3^\infty}$ is not surjective if and only if $\rho_{E,3}$ is not surjective or

$$j_E = -\frac{3^7(t^2-1)^3(t^6+3t^5+6t^4+t^3-3t^2+12t+16)^3(2t^3+3t^2-3t-5)}{(t^3-3t-1)^9}$$

for some $t \in \mathbb{Q}$.

- (iii) If $\ell \geq 5$, then ρ_{E,ℓ^∞} is not surjective if and only if $\rho_{E,\ell}$ is not surjective.

Proof. For the 2-adic and 3-adic cases, see [DD12] and [Elk06], respectively. When $\ell \geq 5$, the lemma follows from Lemma 3.4 of [Ser98, IV §3.4]. \square

APPENDIX A. SOME CODE

Given a non-CM elliptic curve E/\mathbb{Q} , the following Magma function outputs a finite set of primes S such that the representation $\rho_{E,\ell}$ is surjective for all primes $\ell \notin S$. It uses the algorithm of §1.1 if j_E is an integer and uses §1.2 otherwise. (Note that we could then use §6 to quickly determine the minimal such set S)

```

ExceptionalSet:=function(E)
  j:=jInvariant(E); den:=Denominator(j);
  S:={2,3,5,7,13};
  if j in {-11^2,-11*131^3} then S:=S join {11}; end if;
  if j in {-297756989/2, -882216989/131072} then S:=S join {17}; end if;
  if j in {-9317, -162677523113838677} then S:=S join {37}; end if;
  if den ne 1 then
    ispow,b,e:=IsPower(den);
    if ispow then
      P:={p: p in PrimeDivisors(e) | p ge 11};
      if P ne {} then
        g:=GCD({&*P} join {p^2-1 : p in PrimeDivisors(b)});
        S:= S join {ell : ell in PrimeDivisors(g) | ell ge 11};
      end if;
    end if;
  else
    D:=Discriminant(E);
    Q:=PrimeDivisors(GCD(Numerator(j-1728),Numerator(D)*Denominator(D)));
    Q:=[q: q in Q | q ne 2 and IsOdd(Valuation(j-1728,q))];
    if Valuation(j,2) in {3,6,9} then Q:=[2] cat Q; end if;
    p:=2;
    alpha:=[]; beta:=[];
    repeat
      a:=0;
      while a eq 0 do
        p:=NextPrime(p); K:=KodairaSymbol(E,p);
        if K eq KodairaSymbol("I0") then
          a:=TraceOfFrobenius(E,p);
        elif K eq KodairaSymbol("I0*") then
          a:=TraceOfFrobenius(QuadraticTwist(E,p),p);
        end if;
      end while;
    repeat
  end function;

```

```

    end while;
    S:=S join {ell : ell in PrimeDivisors(a) | ell gt 13};
    alpha:= alpha cat [[(1-KroneckerSymbol(q,p)) div 2 : q in Q]];
    beta:= beta cat [ [(1-KroneckerSymbol(-1,p)) div 2] ];
    A:=Matrix(GF(2),alpha); b:=Matrix(GF(2),beta);
    until IsConsistent(Transpose(A),Transpose(b)) eq false;
  end if;
  return S;
end function;

```

The following code verifies Conjecture 1.1 for all elliptic curves E/\mathbb{Q} in Cremona's database [Cre]; currently this includes all curves of conductor at most 360000.

```

D:=CremonaDatabase(); LargestConductor(D);
for N in [1..LargestConductor(D)] do
for E in EllipticCurves(D,N) do
if not HasComplexMultiplication(E) then
  S:={p: p in ExceptionalSet(E) | p gt 13};
  if jInvariant(E) in {-297756989/2, -882216989/131072} then
    assert S eq {17};
  elif jInvariant(E) in {-9317, -162677523113838677} then
    assert S eq {37};
  else
    assert S eq {};
  end if;
end if;
end for;
end for;

```

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