§8.3 (Trig Substitution)
§8.5 (Partial Fractions)
12 July 2018

Name: Solutions

Evaluate the integral using any technique we have learned so far.

(1) $\displaystyle\int \frac{1}{1+x^2}\,dx$

Solution: Let $x = \tan(\theta)$. Then $dx = \sec^2(\theta)\,d\theta$, and

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int d\theta = \theta + C = \arctan(x) + C.$$

(2) $\displaystyle\int \ln(x)\,dx$

Solution: Integration by parts, with $u = \ln(x)$ and $dv = dx$.

(3) $\displaystyle\int \sqrt{4x^2-1}\,dx$

Solution: Trig substitution, with $x = \frac{1}{2}\sec\theta$.

(4) $\displaystyle\int \frac{x}{\sqrt{12-6x-x^2}}\,dx$

Solution: Complete the square under the radical, $12-6x-x^2 = 21-(x+3)^2$, and then substitute $u = x+3$.

(5) $\displaystyle\int \sin^3(x)\cos^3(x)\,dx$

Solution: Rewrite $\sin^3(x) = (1-\cos^2(x))\sin(x)$, and let $u = \cos(x)$.

(6) $\int x \sec^2(x)\, dx$

SOLUTION: Use integration by parts, with $u = x$ and $dv = \sec^2(x)\, dx$.

(7) $\int \frac{1}{\sqrt{9 - x^2}}\, dx$.

SOLUTION: Either substitute $u = 3x$ and use the formula for the derivative of $\sin^{-1}(u)$, or substitute $x = 3\sin\theta$.

(8) $\int x^2 \sqrt{x+1}\, dx$

SOLUTION: Make the substitution $u = x + 1$. Then $du = dx$ and $x^2 = (u-1)^2 = u^2 - 2u + 1$.

(9) $\int \frac{1}{(x+1)(x+2)^3}\, dx$

SOLUTION: Use partial fractions to decompose

$$\frac{1}{(x+1)(x+2)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}.$$

(10) $\int \frac{1}{(x+12)^4}\, dx$

SOLUTION: Substitute $u = x + 12$.

(11) $\displaystyle\int \frac{1}{\sqrt{x^2+9}}\,dx$

SOLUTION: Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$, and $x^2-9 = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta$, so we have

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \boxed{\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| + C}$$

(12) $\displaystyle\int x\sqrt{x^2-5}\,dx.$

SOLUTION: Substitute $u = x^2-5$, so then

$$\int x\sqrt{x^2-5}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}u^{3/2} + C = \boxed{\frac{1}{3}(x^2-5)^{3/2} + C}$$

(13) $\displaystyle\int \frac{3x+5}{x^2-4x-5}\,dx$

SOLUTION: Factor the denominator as $x^2-4x-5 = (x+1)(x-5)$. So we're trying to do partial fractions with

$$\frac{3x+5}{x^2-4x-5} = \frac{3x+5}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x-1}.$$

Clearing denominators, we have

$$3x+5 = A(x+1) + B(x-5).$$

Set $x = 5$ to get $A = \frac{10}{3}$. Set $x = -1$ to get $B = -\frac{1}{3}$. Then we have

$$\frac{3x+5}{x^2-4x-5} = \frac{\frac{10}{3}}{x-5} + \frac{-\frac{1}{3}}{x+1}.$$

Therefore,

$$\int \frac{3x+5}{x^2-4x-5}\,dx = \frac{10}{3}\int \frac{1}{(x-5)}\,dx - \frac{1}{3}\int \frac{1}{x+1}\,dx = \boxed{\frac{10}{3}\ln|x-5| - \frac{1}{3}\ln|x+1| + C}.$$

(14) $\int e^{2x}\cos(x)\,dx$

SOLUTION: Use integration by parts with $u = e^{2x}$ and $dv = \cos(x)\,dx$. Then

$$\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) - \int 2e^{2x}\sin(x)\,dx.$$

Do integration by parts again, this time with $u = e^{2x}$ and $dv = \sin(x)\,dx$. So we have

$$\begin{aligned}\int e^{2x}\cos x\,dx &= e^{2x}\sin(x) - \int 2e^{2x}\sin(x)\,dx \\ &= e^{2x}\sin(x) - 2\left(-e^{2x}\cos(x) - \int(-\cos(x))2e^{2x}\,dx\right) \\ &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,dx\end{aligned}$$

Now add $4\int e^{2x}\cos(x)\,dx$ to both sides, so we have

$$5\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) + 2e^{2x}\cos(x) + C$$

Divide both sides by 5 to get the answer,

$$\boxed{\frac{1}{5}e^{2x}\sin(x) + \frac{2}{5}e^{2x}\cos(x) + C}$$

(15) $\int \cos^2\theta\sin^2\theta\,d\theta$

SOLUTION: First use the identity $\cos^2\theta = 1 - \sin^2\theta$ to write

$$\int \cos^2\theta\sin^2\theta\,d\theta = \int(1-\sin^2\theta)\sin^2\theta\,d\theta = \int\sin^2\theta\,d\theta - \int\sin^4\theta\,d\theta.$$

Using the reduction formula for $\sin^m(x)$,

$$\begin{aligned}\int\cos^2\theta\sin^2\theta\,d\theta &= \int\sin^2\theta\,d\theta - \left(-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int\sin^2\theta\,d\theta\right) \\ &= \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\int\sin^2\theta\,d\theta \\ &= \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\int d\theta\right) \\ &= \boxed{\frac{1}{4}\sin^3\theta\cos\theta - \frac{1}{8}\sin\theta\cos\theta + \frac{1}{8}\theta + C}\end{aligned}$$

(16) $\displaystyle\int \cos(x)\sin^5(x)\,dx$

SOLUTION: Substitute $u = \sin x$, $du = \cos(x)\,dx$.

$$\int \cos(x)\sin^5(x)\,dx = \int u^5\,du = \frac{u^6}{6} + C = \boxed{\frac{\sin^6(x)}{6} + C.}$$

(17) $\displaystyle\int \frac{1}{x(x-1)^2}\,dx$

SOLUTION: Use partial fractions to write

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing denominators gives

$$1 = A(x-1)^2 + Bx(x-1) + Cx.$$

Setting $x = 0$ gives $A = 1$; setting $x = 1$ gives $C = 1$ and setting $x = 2$ gives $B = -1$. The result is

$$\frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}.$$

Now we can integrate.

$$\int \frac{1}{x(x-1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx = \boxed{\ln|x| - \ln|x-1| - \frac{1}{x-1} + C.}$$

(18) $\displaystyle\int \cos^2(4x)\,dx$

SOLUTION: Use the substitution $u = 4x$ and $du = 4\,dx$. Then

$$\begin{aligned}\int \cos^2(4x)\,dx &= \frac{1}{4}\int \cos^2(u)\,du \\ &= \frac{1}{4}\left(\frac{1}{2}u + \frac{1}{2}\sin(u)\cos(u)\right) + C \\ &= \boxed{\frac{1}{2}x + \frac{1}{8}\sin(4x)\cos(4x) + C}\end{aligned}$$

(19) $\displaystyle\int \frac{3}{(x+1)(x^2+x)}\,dx$

SOLUTION: Do partial fractions

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{(x+1)(x)(x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Clearing denominators gives $3 = A(x+1)^2 + Bx(x+1) + Cx$, and setting $x = 0$ give $A = 3$; setting $x = -1$ give $C = -3$. Now plug in $A = 3$ and $C = -3$ to get

$$3 = 3(x+1)^2 + Bx(x+1) - 3x$$

Then set $x = 1$ to get $B = -3$. Therefore,

$$\begin{aligned}\int \frac{3}{(x+1)(x^2+x)}\,dx &= 3\int \frac{1}{x}\,dx - 3\int \frac{1}{x+1}\,dx - 3\int \frac{1}{(x+1)^2}\,dx \\ &= \boxed{3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + C}\end{aligned}$$

(20) $\displaystyle\int (\ln x + 1)\sqrt{(x\ln x)^2 + 1}\,dx$

SOLUTION: Let $u = x\ln x$. Then $du = (1 + \ln x)\,dx$, and

$$\int (\ln x + 1)\sqrt{(x\ln x)^2 + 1}\,dx = \int \sqrt{u^2+1}\,du.$$

Then substitute $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $u^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$. Therefore,

$$\int \sqrt{u^2+1}\,du = \int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

Substitute back $\tan\theta = u$ and $\sec\theta = \sqrt{u^2+1}$, so

$$\int \sqrt{u^2+1}\,du = \frac{1}{2}u\sqrt{u^2+1} + \frac{1}{2}\ln|u + \sqrt{u^2+1}| + C.$$

Finally substitute back $u = x\ln x$.

$$\boxed{\frac{1}{2}x\ln x\sqrt{(x\ln x)^2+1} + \frac{1}{2}\ln\left|x\ln x + \sqrt{(x\ln x)^2+1}\right|}$$

Fun fact: Mathematica wouldn't do this integral for me, but I could do it by hand!