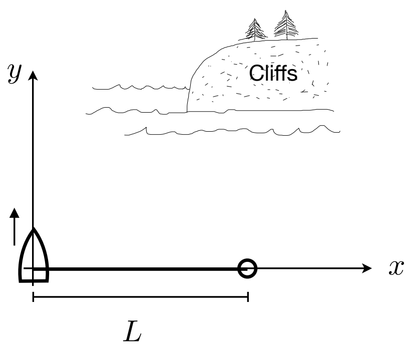
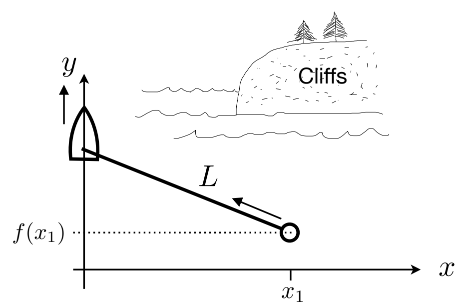
**Introduction:** This workshop gives a glimpse of how calculus pops up in all sorts of unexpected places. Here we travel back in time to the carefree days of summer. Imagine: you're at a summer party hosted by friends at their lake house, and a group of you decide to take the motorboat and go tubing. Fatima claims the tube first and swims off to the end of the rope while you all get settled on the boat, and then off you go! You're cruising along when you notice that Pablo is driving the boat rather close to an approaching cliff face.




Will Fatima have to let go of the rope or will she make it past the rocky cliffs safely? To find out, we will look for the path $y = f(x)$ that the inner tube traces out as it follows the motorboat. The boat is assumed to travel along the $y$–axis, as sketched above.

**Goals:**

- Translate a physical situation into a calculus framework.
- Evaluate an integral of a slope to find the underlying function.

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**Problems:**

**a)** Use the assumptions:

- $y = f(x)$ defines the path that the *inner tube* traces over the water;
- the rope remains taut throughout the motion, with constant length $L$;
- the motion of the inner tube is always in the direction of the rope, i.e. at any given moment, the rope is tangential to the tube's path $y = f(x)$;

to find an expression for $\frac{dy}{dx} = f'(x)$. (*Hint:* Start with a rough sketch of $y = f(x)$ and then think about the meaning of "tangential".)

**Solution.** We begin by defining a value $b_1$ as the vertical side of the triangle outlined in the figure above. Then we can use the Pythagorean Theorem to state:

$$L^2 = x_1^2 + b_1^2$$

Since the first assumption states that $L$ is constant, we can consider this for any value of $x$:

$$L^2 = x^2 + (b(x))^2$$

The third assumption tells us that we can write $\frac{dy}{dx} = f'(x)$ as the slope of the rope as seen in the figure on the right. Then for any given $x$:

$$f'(x) = \frac{\Delta y}{\Delta x} = \frac{(-b)}{x}$$

We can then eliminate the unknown value $b(x)$ to get:

$$f'(x) = \frac{-\sqrt{L^2 - x^2}}{x}$$

**b)** Integrate your $f'(x)$ from part **a)** to find the path $y = f(x)$ that the inner tube will follow. Be sure to use the initial conditions to solve for the constant of integration. (*Hint:* Use the identity $\sin^2\theta = 1 - \cos^2\theta$ to simplify the integrand.)

**Solution.**

$$f(x) = \int f'(x)\ dx = -\int \frac{\sqrt{L^2 - x^2}}{x}\ dx$$

Define the trigonometric substitution $x = L\cos\theta$, $dx = -L\sin\theta\ d\theta$, $\sqrt{L^2 - x^2} = L\sin\theta$:

$$f(x) = -\int \frac{L\sin\theta}{L\cos\theta}(-L\sin\theta\ d\theta) = \int \frac{L\sin^2\theta}{\cos\theta}\ d\theta$$

With the trigonometric identity:

$$f(x) = \int \frac{L(1 - \cos^2\theta)}{\cos\theta}\ d\theta = L\int (\sec\theta - \cos\theta)\ d\theta$$

As we saw in the textbook, $\int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C$, so we integrate to get:

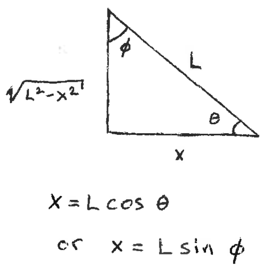
$$f(x) = L\left(\ln|\sec\theta + \tan\theta| - \sin\theta\right) + C$$

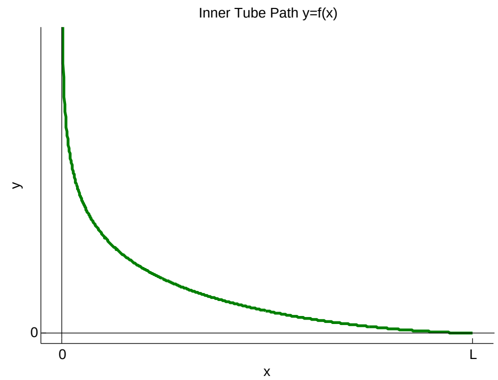
We now transform back from $\theta$ to $x$ with the help of the triangle:

$$f(x) = L\left(\ln\left|\frac{L}{x} + \frac{\sqrt{L^2 - x^2}}{x}\right| - \frac{\sqrt{L^2 - x^2}}{L}\right) + C$$

We know that the inner tube began at $(L, 0)$, so we find $f(L) = L(\ln|1 - 0| - 0) + C = C = 0$, yielding:

$$\boxed{f(x) = L\left(\ln\left|\frac{L}{x} + \frac{\sqrt{L^2 - x^2}}{x}\right| - \frac{\sqrt{L^2 - x^2}}{L}\right).}$$




Note that the calculation also could be performed with $x = L\sin\phi$, $dx = L\cos\phi\ d\phi$, $\sqrt{L^2 - x^2} = L\cos\phi$ with the same end result:

$$
\begin{aligned}
f(x) &= -\int \frac{L\cos\phi}{L\sin\phi}(L\cos\phi\ d\phi) = -\int \frac{L\cos^2\phi}{\sin\phi}\ d\phi \\
\text{trigonometric identity:} & \\
&= \int \frac{L(\sin^2\phi - 1)}{\sin\phi}\ d\phi = L\int(\sin\phi - \csc\phi)\ d\phi \\
\text{integral of cosecant:} & \\
&= L(-\cos\phi - \ln|\csc\phi - \cot\phi|) + C \\
\text{transform back to } x: & \\
&= L\left[-\ln\left|\frac{L - \sqrt{L^2 - x^2}}{x}\right| - \frac{\sqrt{L^2 - x^2}}{L}\right] + C \\
\text{properties of} \ln: & \\
&= L\left[\ln\left|\frac{x}{L - \sqrt{L^2 - x^2}}\right| - \frac{\sqrt{L^2 - x^2}}{L}\right] + C \\
\text{multiply argument of ln by} & \\
\frac{L + \sqrt{L^2 - x^2}}{L + \sqrt{L^2 - x^2}} & \\
\text{and simplify}: & \\
&= L\left[\ln\left|\frac{L + \sqrt{L^2 - x^2}}{x}\right| - \frac{\sqrt{L^2 - x^2}}{L}\right] + C
\end{aligned}
$$

This shape is called a *tractrix*. There are other valid ways of writing the same function. That could include using the rules of logarithms to rearrange or rewrite the first term. That term is also equivalent to $L\,\text{arcsech}\,\frac{x}{L}$.

**c)** Given the path found in part **b)**, and some curve $y = g(x)$ that describes the cliff edge, briefly explain how you would find out if Fatima makes it past the cliffs safely.

**Solution.** We would look for intersections of Fatima's path $y = f(x)$ with the cliff edge $y = g(x)$. That is, if there is an $x$ such that $f(x) = g(x)$, Fatima needs to let go before she gets roughed up by the rocks!

**d)** Are the assumptions in part **a)** reasonable? What physical effects do we ignore? Does the speed of the boat affect our model?

**Partial Solution.** Realistically, while the rope will not stretch, it might become less than taut, depending upon the waves, which are the first clear omission from this model. The speed of the boat (and therefore of Fatima on the inner tube) is not required for this model at all, but would certainly affect the actual experience! One might even consider the weight of Fatima to be a parameter in this problem, since it could determine whether the inner tube is lightly skimming the surface or being dragged back by the friction of the water around her.

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**Workshop takeaways:**

- Trig substitution can be used in multiple ways to solve the same integral.
- The same function can be written in multiple equivalent ways (that don't necessarily look all that equivalent!).
- Simple physical assumptions can be translated into "math".

§9.1 (ARC LENGTH & SURFACE AREA) NAME: SOLUTIONS

17 July 2018

The **arc length** of the graph of $y = f(x)$ over the interval $[a, b]$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

The **surface area** of the solid of revolution $S$ formed by rotating the graph of $y = f(x)$ about the $x$-axis over the interval $[a, b]$ is

$$\int_a^b 2\pi|f(x)|\sqrt{1 + f'(x)^2}\, dx.$$

Compute the surface area of revolution about the $x$-axis over the given interval.

(1) $y = (4 - x^{2/3})^{3/2}$, $[0, 8]$

SOLUTION:

$$f'(x) = \frac{3}{2}\sqrt{4 - x^{2/3}} \cdot \left(-\frac{2}{3}\right) x^{-1/3} = \frac{-\sqrt{4 - x^{2/3}}}{x^{1/3}}$$

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{4 - x^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + 4 - x^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}$$

So the surface area is:

$$2\pi \int_0^8 (4 - x^{2/3})^{3/2} \cdot \frac{2}{x^{1/3}}\, dx.$$

To solve this, let $u = 4 - x^{2/3}$, so $du = -\frac{2}{3x^{1/3}}$. The integral becomes

$$2\pi \int_4^0 u^{3/2}(-3)\, du = \frac{384\pi}{5}.$$

(2) $y = e^{-x}$, $[0, 1]$

SOLUTION: $f'(x) = -e^{-x}$ and the surface area is

$$2\pi \int_0^1 e^{-x}\sqrt{1 + e^{-2x}}\, dx.$$

To integrate, substitute $u = e^{-x}$, $du = e^{-x}\, dx$. We will evaluate the indefinite integral and find an antiderivative, then plug in the bounds.

$$2\pi \int e^{-x}\sqrt{1 + e^{-2x}}\, dx = 2\pi \int \sqrt{1 + u^2}\, du.$$

Substitute $u = \tan\theta$, $du = \sec^2\theta\, d\theta$, so we have

$$\begin{aligned} 2\pi \int \sqrt{1 + u^2}\, du &= 2\pi \int \sec^2\theta\, d\theta \\ &= -\pi \sec\theta \tan\theta - \pi \ln|\sec\theta + \tan\theta| + C \\ &= -\pi e^{-x}\sqrt{1 + e^{-2x}} - \pi \ln|\sqrt{1 + e^{-2x}} + e^{-x}| \end{aligned}$$

Evaluating from $x = 0$ to $x = 1$ gives a surface area of

$$\pi\sqrt{2} - \pi e^{-1}\sqrt{1 + e^{-2}} + \pi \ln\left(\frac{\sqrt{2} + 1}{\sqrt{1 + e^{-2}} + e^{-1}}\right)$$

(3) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad [1, e]$

SOLUTION: We have $y' = \frac{x}{2} - \frac{1}{2x}$, and

$$1 + (y')^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

Therefore,

$$\begin{aligned}
SA &= 2\pi \int_1^e \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^e \left(\frac{x^3}{8} + \frac{x}{8} - \frac{x\ln x}{4} - \frac{\ln x}{4x}\right) dx \\
&= 2\pi \left(\frac{x^4}{32} + \frac{x^2}{16} - \frac{x^2 \ln x}{8} + \frac{x^2}{16} - \frac{(\ln x)^2}{8}\right)\Bigg|_1^e \\
&= 2\pi \left(\frac{e^4}{32} + \frac{e^2}{16} - \frac{e^2}{8} + \frac{e^2}{16} - \frac{1}{8} - \left(\frac{1}{32} + \frac{1}{16} + 0 + \frac{1}{16} - 0\right)\right) \\
&= \frac{\pi}{16}\left(e^4 - 9\right)
\end{aligned}$$

(4) $y = \cos(x), \quad [0, \pi]$

SOLUTION: Notice that $\cos(x)$ is negative on the interval from $\pi/2$ to $\pi$, so we need to be careful that we don't pick up a negative sign in the integral! Therefore, we should split the integral at $x = \pi/2$

$$SA = \int_0^{\pi/2} 2\pi \cos(x)\sqrt{1 + \sin^2(x)}\, dx + \int_{\pi/2}^{\pi} 2\pi(-\cos(x))\sqrt{1 + \sin^2(x)}\, dx.$$

However, this solid of revolution is an hourglass shape, and both the top and bottom of this shape have the same surface area. So we can compute one of these integrals and double the answer.

$$SA = 2\int_0^{\pi/2} 2\pi \cos(x)\sqrt{1 + \sin^2(x)}\, dx.$$

To compute this integral, let $u = \sin(x)$, $du = \cos(x)\, dx$. Then

$$SA = 4\pi \int_0^1 \sqrt{1 + u^2}\, du.$$

Now substitute $u = \tan\theta$, $du = \sec^2\theta\, d\theta$.

$$\begin{aligned}
SA &= 4\pi \int_0^{\pi/4} \sec^3\theta\, d\theta \\
&= -2\pi \sec\theta \tan\theta - 2\pi \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} = 2\pi\left(\sqrt{2} + \ln(\sqrt{2} + 1)\right)
\end{aligned}$$