§8.8 (PROBABILITY) NAME: SOLUTIONS

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RANDOM VARIABLES AND PROBABILITY DENSITY FUNCTIONS

A **random variable** X is a variable whose possible values are outcomes of a random phenomena. The probability that the outcome of a random trial lies in a range (a, b) is determined by its **probability density function** $p(x)$:

$$P(a \leq X \leq b) = \boxed{\int_a^b p(x)\, dx} \quad (1) \quad .$$

There are two requirements for a function to be a probability density function on a domain (a, b):

(i) $\boxed{p(x) \geq 0 \text{ for all } x}$ (2)

(ii) $\boxed{\int_a^b p(x)\, dx = 1}$ (3)

QUESTIONS

(1) If X is a random variable distributed according to $p(x)$ over $(-\infty, \infty)$, then write integrals to represent the following probabilities:

(a) $P(-4 \leq X \leq 1)$

SOLUTION: $P(-4 \leq X \leq 1) = \int_{-4}^{1} p(x)\, dx$

(b) $P(X \geq 2)$

SOLUTION: $P(X \geq 2) = \int_2^\infty p(x)\, dx$

(c) $P(X = 0)$

SOLUTION: The probability that X is *exactly* zero is zero!

$$P(X = 0) = \int_0^0 p(x)\, dx.$$

There's no chance that

(d) The probability that X is in the interval $[-\infty, 0) \cup (1, \infty)$.

SOLUTION:

$$\int_{-\infty}^{0} p(x)\, dx + \int_1^\infty p(x)\, dx.$$

(2) If instead the domain of $p(x)$ in the previous question is $(-\infty, 7)$, how should you modify your answers to the previous questions?

SOLUTION: The first and third answers can stay the same, but the second and fourth should change to include the new upper bound:

$$P(X \geq 2) = \int_2^7 p(x)\,dx.$$

$$P(-\infty \leq X \leq 0 \text{ or } 1 \leq X \leq 7) = \int_{-\infty}^0 p(x)\,dx + \int_1^7 p(x)\,dx.$$

(3) Find a constant C such that $p(x) = \dfrac{C}{(2+x)^3}$ is a probability density function on the interval $[2, 4]$.

SOLUTION: For $p(x)$ to be a probability density function, it must integrate to 1 over the given integral. So

$$1 = \int_2^4 p(x)\,dx = \int_2^4 \frac{C}{(2+x)^3}\,dx = \left.\frac{-C}{2(2+x)^2}\right|_2^4 = C\left(\frac{-1}{72} + \frac{1}{32}\right) = C\frac{5}{288}$$

Therefore, $\boxed{C = \dfrac{288}{5}}$.

(4) Verify that $\dfrac{5}{4x^2}$ is a probability density function on $[1, 5]$ and if X is the corresponding random variable, find $P(1 \leq X \leq 3)$.

SOLUTION:

$$\int_1^5 \frac{5}{4x^2}\,dx = \frac{5}{4}\left.\left(\frac{-1}{x}\right)\right|_1^5 = \frac{5}{4}\left(\frac{-1}{5} + 1\right) = \frac{5}{4}\frac{4}{5} = 1.$$

$$P(1 \leq X \leq 3) = \int_1^3 \frac{5}{4x^2}\,dx = \frac{5}{4}\frac{2}{3} = \frac{5}{6}.$$

Mean, Standard Deviation, Exponential Densities

The **mean** or **average** or **expected value** of a random variable X with PDF $p(x)$ over (a, b) is

$$\mu(X) = \boxed{\int_a^b xp(x)\,dx.} \quad (4)$$

The **exponential distribution** with mean r is the probability density function on $[0, \infty)$ defined by

$$p(t) = \boxed{\frac{1}{r}e^{-t/r}.} \quad (5)$$

The **standard deviation** of a random variable X with probability density function $p(x)$ on (a, b) and mean μ is

$$\sigma(x) = \int_a^b (x-\mu)^2 p(x)\,dx.$$

Questions

(5) Calculate the mean of the probability density function $\frac{5}{4x^2}$ over $[1, 5]$.

SOLUTION:

$$\int_1^5 x\frac{5}{4x^2}\,dx = \frac{5}{4}\int_1^5 \frac{1}{x}\,dx = \frac{5}{4}\ln(5)$$

(6) The time between incoming calls at a call center is a random variable with exponential density. There is a 50% probability of waiting 20 seconds or more between calls. What is the average time between calls?

SOLUTION: Let X be the random variable for time between calls.

$$P(0 \le X \le 20) = 0.5$$

We know that X is distributed according to $\frac{1}{r}e^{-t/r}$, where r is the average time between calls.

$$0.5 = P(0 \le X \le 20) = \int_0^{20} \frac{1}{r}e^{-t/r}\,dt = \frac{1}{r}\left(-re^{-t/r}\right)\Big|_0^{20} = -e^{-20/r} + 1$$

Then solve for r:

$$1 - e^{-20/r} = 0.5 \implies e^{-20/r} = 0.5 \implies \frac{-20}{r} = \ln(0.5) = -\ln(2) \implies r = \frac{20}{\ln(2)}.$$

(7) The probability density function for the amount of time that elapses before a fire spreads through a forest (once a fire is started) is $p(x) = Cx^{-4}$, defined on $[1, \infty)$, where C is some unknown constant and x is measured in minutes.

(a) Determine C so that $p(x)$ is a probability density function on $[1, \infty)$.

SOLUTION: $1 = \int_1^\infty Cx^{-4}dx = \lim_{R\to\infty}\int_1^R Cx^{-4}dx = \lim_{R\to\infty}\frac{-C}{3}(R^{-3} - 1^{-3}) = \frac{C}{3}$, so $C = 3$.

Note that $p(x) = 3x^{-4} \geq 0$ so this function satisfies both conditions.

(b) Find its mean value and standard deviation.

SOLUTION: $\mu = \int_1^\infty x(3x^{-4})dx = \lim_{R\to\infty}\int_1^R 3x^{-3}dx = \frac{3}{2}$

$\sigma^2 = \int_1^\infty \left(x - \frac{3}{2}\right)^2 3x^{-4}dx = \lim_{R\to\infty}\int_1^R \left(x^2 - 3x + \frac{9}{4}\right) 3x^{-3}dx = \frac{3}{4}$, so $\sigma = \frac{\sqrt{3}}{2}$

(c) Compute the probability that the fire has spread between minutes 2 and 3.

SOLUTION: $P(2 \leq X \leq 3) = \int_2^3 3x^{-4}dx = \frac{117}{1000}$.

(d) How long do we need to wait before there is a 70% probability that the fire has spread?

SOLUTION: $\int_1^t 3x^{-4}dx = -x^{-3}|_1^t = -\frac{1}{t^3} + 1 = 0.70$ so $0.30 = \frac{1}{t^3}$ and $t = \sqrt[3]{1/0.3} \simeq 1.5$min

The Normal Distribution

The **standard normal distribution** is the probability density function

$$p(x) = \boxed{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}.} \tag{6}$$

The normal distribution with mean μ and standard deviation σ is

$$p(x) = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}} \tag{7}$$

To calculate probabilities for the normal distribution, we use the **standard normal cumulative density function**

$$F(z) = \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}} \tag{8}$$

Theorem 1. *If X has a normal distribution with mean μ and standard deviation σ, then for all $a \leq b$,*

$$P(X \leq b) = F\left(\frac{b-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

Questions

(8) A company produces boxes that are filled on average with 16 oz of halloween candy. Due to the witch's curse placed on the founder of the company many generations ago, the actual volume of candy is normally distributed with a standard deviation of 0.4 oz. Find $P(X > 17)$, the probability of a box having more than 17 oz of candy. It may be helpful to know that $F(2.5) \approx 0.99379$.

SOLUTION: The problem tells us that the mean is $\mu = 16$ and the standard deviation of $\sigma = 0.4$. We need to find $P(X > 17)$. We have

$$\begin{aligned}
P(X > 17) &= 1 - P(X \leq 17) \\
&= 1 - F\left(\frac{17-16}{0.4}\right) \\
&= 1 - F(2.5) \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2.5} e^{-t^2/2}\, dt \\
&= 1 - 0.99379 \\
&\approx 0.00621
\end{aligned}$$