§11.4 (ALTERNATING SERIES) NAME: SOLUTIONS

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ABSOLUTE AND CONDITIONAL CONVERGENCE

- **Absolute Convergence:** A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.
- **Absolute Convergence Theorem:** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- **Conditional Convergence:** A series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.
- **Alternating Series Test:** If the sequence $\{b_n\}$ is positive and decreasing, and $\lim_{n\to\infty} b_n = 0$, then $S = \sum_{n=1}^{\infty} (-1)^n b_n$ converges. Furthermore, the partial sums satisfy $|S - S_N| < b_{N+1}$.

PROBLEMS

(1) Show that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges conditionally.

SOLUTION: We first show that the series converges, using the Alternating series test. The terms $a_n = \frac{n}{n^2+1}$ tend to zero since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Moreover, a_n is a decreasing sequence because $f(x) = \frac{x}{x^2+1}$ is decreasing for $x \geq 1$. Therefore, the alternating series test applies and the series converges.

However, to show conditional convergence, we have to show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges. We can do this with the limit comparison test, comparing a_n to $1/n$. We have

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Therefore, because the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

(2) Does $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$ converges absolutely, conditionally, or not at all?

SOLUTION: Compute the limit

$$\lim_{n\to\infty} \frac{n^4}{n^3+1} = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n^3}} = \infty.$$

It follows that the general term $\frac{(-1)^n n^4}{n^3+1}$ of the series doesn't tend to zero, hence this series diverges by the divergence test.

(3) Consider the series $\sum_{n=2}^{\infty} \frac{\cos n\pi}{(\ln n)^2}$.

(a) Show that the series doesn't converge absolutely by using the Direct Comparison Test.

SOLUTION: $\left|\frac{\cos(n\pi)}{(\ln n)^2}\right| = \frac{1}{(\ln n)^2} > \frac{1}{n}$ for every $n \geq 2$

(b) Does it converge conditionally?

SOLUTION: Yes; rewrite the series as $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{(\ln n)^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^2}$ and use the Leibniz Test.

(4) Find a value of N such that the N-th partial sum S_N approximates the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)(n+3)}$ with an error of at most 10^{-5} (calculator needed).

SOLUTION: Using the fact that $|S - S_N| \le |a_{N+1}|$, we see that it suffices to find an N such that $|a_{N+1}| \le 10^{-5}$.

$$|a_{N+1}| = \frac{1}{(N+1)(N+1+2)(N+1+3)} = \frac{1}{(N+1)(N+3)(N+4)} \le 10^{-5}$$

is equivalent to asking

$$(N+1)(N+3)(N+4) \ge 10^5$$

Trying with some values of N, we see that the minimum value that makes this true is $N = 44$.

(5) Prove that $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ and $\displaystyle\lim_{n\to 0} \sqrt[n]{1+n} = e$.

SOLUTION: This can be done using L'Hôpital's rule. We will do the first one; the second is just a change of variables of the first.

Let $y = \left(1 + \frac{1}{n}\right)^n$. Then

$$\ln y = n \ln\left(1 + \frac{1}{n}\right).$$

Then,

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right).$$

This is an indeterminate form, $\infty \cdot 0$. So we manipulate it so we can use L'Hôpital's rule.

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(1 + \frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1$$

Of course, this is $\lim_{n\to\infty} \ln y$. We were looking for $\lim_{n\to\infty} y$. But we can solve for y – since $\ln(x)$ is a continuous function, $\lim_{n\to\infty} \ln y = \ln\left(\lim_{n\to\infty} y\right)$. Hence,

$$\ln\left(\lim_{n\to\infty} y\right) = 1 \implies \lim_{n\to\infty} y = e.$$