HOMEWORK 2 NAME: SOLUTIONS

Math 1910, Summer 2018 Due 12 July 2018

Applying a force to an object causes a change in linear momentum. This is Newton's Second Law: $F = ma$. This equation may also be written as

$$F = ma = \frac{d}{dt}mv = \frac{dP}{dt},$$

where P is the linear momentum of the object.

Similarly, applying a *torque* (angular force) to an object causes a change in angular momentum. The equivalent expression of Newton's Second Law for rotational motion is

$$T = I\alpha = \frac{d}{dt}I\omega = \frac{dH}{dt},$$

where T is the torque, $H = I\omega$ is angular momentum, α is angular acceleration, ω is angular velocity, and I is the *moment of inertia*.

The moment of inertia is computed relative to the axis of rotation: it measures how difficult it is to change the angular velocity of the object, similar to how mass dictates how difficult it is to change the linear velocity. The moment of inertia of a mass M rotating at a distance r from the axis is Mr^2 . This holds even if the mass is distributed over, say, a cylinder (empty, with no top or bottom, and with its thickness very small relative to r).

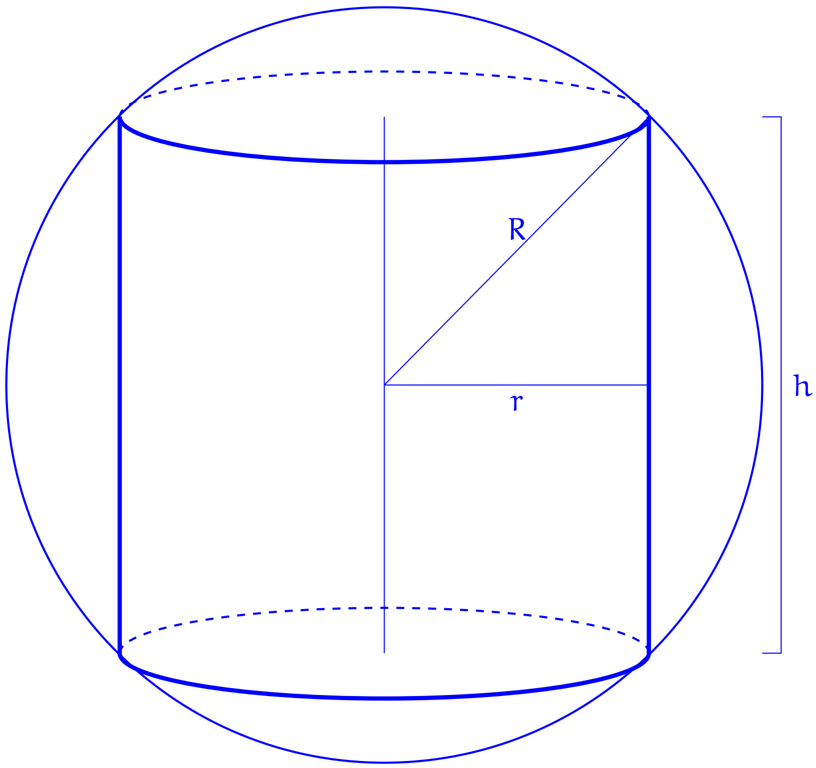
If you have a solid object, not all of the mass is at the same distance from the axis of rotation. In this case, to find I you must use an integral. If the mass of an object is distributed between distances a and b from the axis, and the differential dM describes a little chunk of mass a distance r from the rotation axis, then the moment of inertia is

$$I = \int_a^b r^2 dM. \tag{$\star$}$$

This homework will lead you towards calculating the angular momentum of the Earth. We will model the Earth as a solid uniform sphere with a mass density of $\rho = 5500$ kg/m^3 that rotates about its north-south axis once per day. The average radius of the Earth is $R = 6378$ km.

(1) Consider a uniform solid sphere of outer radius R and constant density ρ spinning about its vertical axis. Slice it into concentric cylindrical shells, each of thickness dr. Sketch and label the dimensions of one of these cylindrical shells.

SOLUTION:



(2) Find an expression for dM, the mass of a slice with thickness dr located a distance r away from the rotation axis.

SOLUTION: A cylindrical shell of thickness dr with radius r contained in a sphere of radius R has height $h = 2\sqrt{R^2 - r^2}$ and circumference $2\pi r$. Thus,

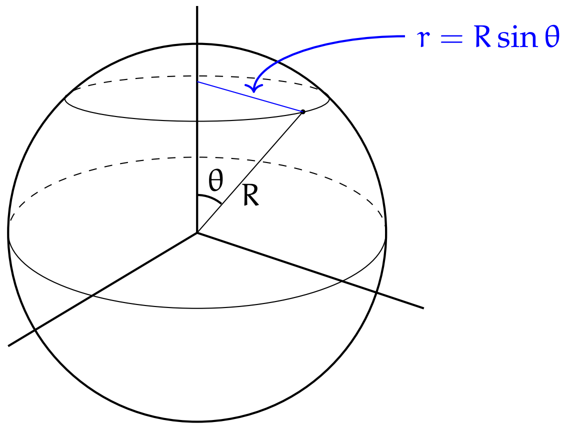
$$dM = \rho \, dV = \rho(2\pi r)h \, dr = \rho(2\pi r)(2\sqrt{R^2 - r^2}) \, dr = 4\pi r\rho\sqrt{R^2 - r^2} \, dr.$$

(3) Set up an integral to compute the moment of inertia I_E of the Earth using the formula $I = \int r^2 \, dM$.

SOLUTION:

$$I = \int r^2 dM = \int_0^R r^2 4\pi r\rho\sqrt{R^2 - r^2} \, dr = 4\pi\rho \int_0^R r^3 \sqrt{R^2 - r^2} \, dr.$$

(4) Let θ denote the *zenith angle* measured from the north pole (see figure). A point on the surface of the sphere at angle θ is at distance $r = R\sin(\theta)$ away from the axis. Use the substitution $r = R\sin(\theta)$ to evaluate the integral. (*Hint:* $\sin^2(\theta) = 1 - \cos^2(\theta)$)



SOLUTION: Because $r = R\sin\theta$, $dr = R\cos(\theta)\,d\theta$. For r to vary between 0 and R, θ must vary from 0 to $\pi/2$. Substituting into the previous integral, we obtain

$$I = 4\pi\rho\int_0^{\pi/2}(R^3\sin^3\theta)\sqrt{R^2 - R^2\sin^2\theta}\,(R\cos\theta)\,d\theta = 4\pi\rho R^5\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

Now substitute $\sin^2\theta = 1 - \cos^2\theta$.

$$\begin{aligned}
I = 4\pi\rho R^5\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta &= 4\pi\rho R^5\int_0^{\pi/2}\sin(\theta)(1-\cos^2\theta)\cos^2\theta\,d\theta \\
&= 4\pi\rho R^5\int_0^{\pi/2}(\cos^2\theta - \cos^4\theta)\sin(\theta)\,d\theta \\
&= 4\pi\rho R^5\left(-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Bigg|_0^{\pi/2} \\
&= 4\pi\rho R^5\frac{1}{15}\left(3\cos^5\theta - 5\cos^3\theta\right)\Bigg|_0^{\pi/2} \\
&= \frac{4}{15}\pi\rho R^5\left((0-0)-(3-5)\right) \\
&= \frac{8}{15}\pi\rho R^5
\end{aligned}$$

(5) Show that your answer to the previous question is equivalent to $I = \frac{2}{5}MR^2$, where M is the total mass of the sphere. What are the units of I?

SOLUTION: The volume of a sphere is $\frac{4}{3}\pi R^3$, so by direct computation we have

$$\frac{2}{5}MR^2 = \frac{2}{5}R^2\left(\frac{4}{3}\pi R^3\right)\rho = \frac{8}{15}\pi\rho R^5,$$

same as the answer to the previous question. The units of I are mass times area, or $\mathrm{kg}\cdot\mathrm{m}^2$.

(6) Find the Earth's angular velocity ω_E in radians per second. SOLUTION: The Earth rotates 2π radians in one day, and one day is 86400 seconds. Hence,

$$\omega_E = \frac{2\pi}{86400\ \mathrm{s}} \approx 7.35 \times 10^{-5}\ \mathrm{rad/s}.$$

(7) Compute the total angular momentum $H_E = I_E\omega_E$ of the Earth.

SOLUTION:

$$H_E = I_E\omega_E = \frac{8\pi}{15}\left(5500\ \mathrm{kg/m^3}\right)\left(6.378 \times 10^6\ \mathrm{m}\right)^5\left(\frac{2\pi}{86400\ \mathrm{s}}\right) = 7.073 \times 10^{33}\ \mathrm{kg}\cdot\mathrm{m^2/s}$$

(8) The actual angular momentum of the Earth is $H_E \approx 8.038 \times 10^{33}$ J s, which is about 13.6% different from the answer to the previous question. What might explain this error?

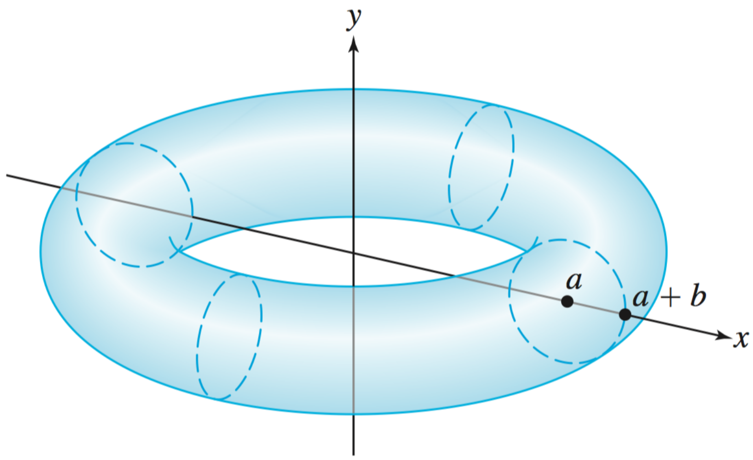
SOLUTION: Some possible reasons:

- The Earth is not a perfect sphere.
- The Earth is not uniformly dense.
- The Earth has oceans and and atmosphere sloshing around on it, as well as a molten mantle that sloshes around as it rotates.
- The Earth doesn't rotate exactly once per day.

There are probably some other reasons as well.

If an object has an angular momentum H, then it takes H/t Joules of energy applied continuously over t seconds to stop that object from rotating. Halting the rotation of the Earth even over the course of several hours would require collision with a Mars-sized asteroid. Such a collision has happened at least once in the past – Earth and a Mars-sized planet called Theia collided 4.4 billion years ago, melting the surface of the Earth, destroying Theia, and tilting the Earth's axis from vertical to 23.5°. The debris from the collision coalesced to form the moon.

(9) A *torus* is the doughnut shape pictured below. It is produced by rotating the circle $(x-a)^2 + y^2 = b^2$ around the y-axis (assuming $a > b$). Calculate the volume of this torus in two different ways:



(a) Using the shell method.

SOLUTION:

When rotating the region enclosed by the circle $(x-a)^2 + y^2 = b^2$ about the y-axis each shell has radius x and height

$$\sqrt{b^2-(x-a)^2} - (-\sqrt{b^2-(x-a)^2}) = 2\sqrt{b^2-(x-a)^2}.$$

The volume of the resulting torus is then

$$2\pi \int_{a-b}^{a+b} 2x\sqrt{b^2-(x-a)^2}dx.$$

Let $u = x - a$. Then $du = dx$, $x = u + a$, and

$$\begin{aligned} 2\pi \int_{a-b}^{a+b} 2x\sqrt{b^2-(x-a)^2}dx &= 2\pi \int_{-b}^{b} 2(u+a)\sqrt{b^2-u^2}du \\ &= 4\pi \int_{-b}^{b} u\sqrt{b^2-u^2}du + 4a\pi \int_{-b}^{b} \sqrt{b^2-u^2}du. \end{aligned}$$

Now

$$\int_{-b}^{b} u\sqrt{b^2-u^2}du = 0$$

because the integrand is an odd function and the integration interval is symmetric with respect to zero. The other integral is one half the area of a circle of radius b, and so we have

$$\int_{-b}^{b} \sqrt{b^2-u^2}du = \frac{1}{2}\pi b^2.$$

Finally, the volume of the torus is $4\pi(0) + 4a\pi(\frac{1}{2}\pi b^2) = 2\pi^2 ab^2$.

(b) Using the disk method.

SOLUTION: Rotating the region enclosed by the circle $(x-a)^2+y^2=b^2$ about the y-axis produces a torus whose cross sections are washers with outer radius $R=a+\sqrt{b^2-y^2}$ and inner radius $r=a-\sqrt{b^2-y^2}$. The volume of the torus is then

$$\int_{-b}^{b}((a+\sqrt{b^2-y^2})^2-(a-\sqrt{b^2-y^2})^2)\,dy = 4a\pi\int_{-b}^{b}\sqrt{b^2-y^2}\,dy.$$

The remaining definite integral is one half the area of a circle of radius b, and so the volume of the torus is

$$4a\pi\cdot\frac{1}{2}\pi b^2 = 2\pi^2 ab^2.$$