1 The Main Example

Consider the dual pair $(Sp(m,\mathbb{R}), O(p,q)) \subset Sp(mn,\mathbb{R})$, with $n = p+q$, and $p \geq q \geq m$. The last condition asserts that we are in the stable case. Let $P = MN$ be the Siegel parabolic subgroup of $Sp(m,\mathbb{R})$. In the theory of the oscillator representation there are very explicit formulas for the action of $P \times O(p,q)$ on $L^2(\mathbb{R}^{mn})$. Of course this action doesn't extend to a representation of $Sp(m,\mathbb{R}) \times O(p,q)$, we would need to go to the double cover to do that, but it does extend to a projective representation of $Sp(m,\mathbb{R}) \times O(p,q)$. To write down this explicit action we will identify $\mathbb{R}^{mn}$ with $Hom(\mathbb{R}^m, \mathbb{R}^n)$, and we will fix a unitary character χ of $\mathbb{R}$. The action is then given by

$$\left(\begin{bmatrix} I & X \\ & I \end{bmatrix} \cdot \varphi \right)(T) = \chi(\operatorname{tr} XT^t I_{p,q} T)\varphi(T) \tag{1}$$

$$\left(\begin{bmatrix} A & \\ & A^{-t} \end{bmatrix} \cdot \varphi \right)(T) = |\det A|^{\frac{n}{2}} \varphi(TA), \qquad A \in GL(m,\mathbb{R}) \tag{2}$$

$$(g \cdot \varphi)(T) = \varphi(g^{-1}T), \qquad g \in O(p,q). \tag{3}$$

Using this formulas we want to describe $L^2(Hom(\mathbb{R}^m, \mathbb{R}^n))$ as a representation of $P \times O(p,q)$. Let

$$U = \left\{ T \in Hom(\mathbb{R}^m, \mathbb{R}^n) \;\middle|\; \begin{array}{c} T \text{ is of maximal rank and the inner} \\ \text{product on } T(\mathbb{R}^m) \text{ is non-degenerated} \end{array} \right\}$$

Observe that $U \subset Hom(\mathbb{R}^m, \mathbb{R}^n)$ is open, dense, and $P \times O(p,q)$-invariant. Let $r, s \geq 0$ be a pair of integers such that $r + s = m$, and define

$$U_{r,s} = \{ T \in U \,|\, T(\mathbb{R}^m) \text{ has signature } (r,s) \}.$$

It's clear that

$$U = \bigcup_{r+s=m} U_{r,s}$$

and the $U_{r,s}$ are the open orbits of the action of $P \times O(p,q)$ on $Hom(\mathbb{R}^m, \mathbb{R}^n)$, hence as a $P \times O(p,q)$-module

$$L^2(Hom(\mathbb{R}^m, \mathbb{R}^n)) \cong \bigoplus_{r+s=m} L^2(U_{r,s}). \tag{4}$$

We will now describe $L^2(U_{r,s})$ as a $P \times O(p,q)$-module. Let $T_{r,s} \in U_{r,s}$ be given by $T_{r,s} e_i = e_{p-r+i}$, and define a character $\chi_{r,s}$ on N by the formula

$$\chi_{r,s}\left(\begin{bmatrix} I & X \\ & I \end{bmatrix} \right) = \chi(\operatorname{tr} XT_{r,s}^t I_{p,q} T_{r,s}) = \chi(\operatorname{tr} X I_{r,s}).$$

Let $H_{r,s}$ be the stabilizer of $T_{r,s}$ in $M \times O(p,q)$. Then

$$\begin{aligned} L^2(U_{r,s}) &\cong \operatorname{Ind}_{H_{r,s}N}^{P\times O(p,q)} \chi_{r,s} \\ &\cong \operatorname{Ind}_{O(r,s)N \times O(r,s) \times O(p-r,q-s)}^{P\times O(p,q)} \operatorname{Ind}_{H_{r,s}N}^{O(r,s)N\times O(r,s)\times O(p-r,q-s)} \chi_{r,s} \\ &\cong \operatorname{Ind}_{O(r,s)N \times O(r,s) \times O(p-r,q-s)}^{P\times O(p,q)} \int_{\hat{O(r,s)}} \tau^* \chi_{r,s} \otimes \tau \otimes 1 \, d\nu(\tau) \\ &\cong \int_{\hat{O(r,s)}} \operatorname{Ind}_{O(r,s)N}^{P} \tau^* \chi_{r,s} \otimes \operatorname{Ind}_{O(r,s)\times O(p-r,q-s)}^{O(p,q)} \tau \otimes 1 \, d\nu(\tau). \end{aligned} \tag{5}$$

Now from (4) and (5)

$$L^2(Hom(\mathbb{R}^m,\mathbb{R}^n)) \cong \bigoplus_{r+s=m} \int_{O(\hat{r,s})} \mathrm{Ind}_{O(r,s)N}^{P} \tau^*\chi_{r,s} \otimes \mathrm{Ind}_{O(r,s)\times O(p-r,q-s)}^{O(p,q)} \tau \otimes 1 \, d\nu(\tau). \quad (6)$$

Now as a representation of $O(r,s)\times O(p,q)$

$$\begin{aligned}
\mathrm{Ind}_{O(p-r,q-s)}^{O(p,q)} 1 &\cong \mathrm{Ind}_{O(r,s)\times O(r,s)\times O(p-r,q-s)}^{O(r,s)\times O(p,q)} \mathrm{Ind}_{\Delta O(r,s)\times O(p-r,q-s)}^{O(r,s)\times O(r,s)\times O(p-r,q-s)} 1\otimes 1 \\
&\cong \mathrm{Ind}_{O(r,s)\times O(r,s)\times O(p-r,q-s)}^{O(r,s)\times O(p,q)} \int_{O(\hat{r,s})} \tau^*\otimes\tau\otimes 1 \, d\nu(\tau) \\
&\cong \int_{O(\hat{r,s})} \tau^*\otimes \mathrm{Ind}_{O(r,s)\times O(p-r,q-s)}^{O(p,q)} \tau\otimes 1\, d\nu(\tau) \\
&\cong \int_{O(\hat{r,s})} \tau^*\otimes \int_{O(\hat{p,q})} M_{\tau,r,s}(\pi)\otimes\pi \, d\tilde{\eta}(\pi)\, d\nu(\tau) \\
&\cong \int_{O(\hat{p,q})}\int_{O(\hat{r,s})} \tau^*\otimes M_{\tau,r,s}(\pi)\otimes\pi \, d\nu(\tau)\, d\tilde{\eta}(\pi). \quad (7)
\end{aligned}$$

On the other hand as a representation of $O(p,q)$ the space $\mathrm{Ind}_{O(p-r,q-s)}^{O(p,q)} 1$ has been described in the work of Delorme, Schlichtkrull and Van Den Ban as

$$\mathrm{Ind}_{O(p-r,q-s)}^{O(p,q)} 1 = \int_{O(\hat{p,q})} M_{r,s}(\pi)\otimes\pi \, d\eta(\pi). \quad (8)$$

From (7) and (8)

$$M_{r,s}(\pi) \cong \int_{O(\hat{r,s})} \tau^*\otimes M_{\tau,r,s}(\pi)\, d\nu(\tau),$$

and $\tilde{\eta}$ is in the same measure class as η. From all this and (6)

$$\begin{aligned}
&L^2(Hom(\mathbb{R}^m,\mathbb{R}^n)) \cong \\
&\bigoplus_{r+s=m} \int_{O(\hat{r,s})} \mathrm{Ind}_{O(r,s)N}^{P} \tau^*\chi_{r,s} \otimes \int_{O(\hat{p,q})} M_{\tau,r,s}(\pi)\otimes\pi \, d\nu(\tau)\, d\eta(\pi) \cong \\
&\bigoplus_{r+s=m} \int_{O(\hat{p,q})}\int_{O(\hat{r,s})} M_{\tau,r,s}(\pi)\otimes \mathrm{Ind}_{O(r,s)N}^{P} \tau^*\chi_{r,s}\otimes\pi \, d\nu(\tau)\, d\eta(\pi). \quad (9)
\end{aligned}$$

On the other hand, the abstract theory of Howe duality establishes that as a $P\times O(p,q)$-module

$$L^2(Hom(\mathbb{R}^m,\mathbb{R}^n)) \cong \int_{Sp(\hat{m,\mathbb{R}})} \pi|_P \otimes \Theta(\pi)\, d\mu(\pi). \quad (10)$$

Since we are working on the stable range, we know that μ is the Plancherel measure for $Sp(m,\mathbb{R})$. The representation $\Theta(\pi)$ has been determined by the work of Jian-Shu Li among others. We are thus left with the problem of decomposing an irreducible tempered representation of $Sp(m,\mathbb{R})$ when restricted to P.

Let G be a Lie group of tube type, and let $P=MN$ be a parabolic subgroup such that N is abelian. We will look at $L^2(G)$ as a $P\times G$-module. We have an isomorphism

$$L^2(G)\cong L^2(\hat{N},E,\eta)$$

with $E_\chi \cong \operatorname{Ind}_N^P End(\chi)$ given in the following way: Given $f \in L^2(G)$, define $s_f \in L^2(\hat{N}, E, \eta)$ by

$$s_f(\chi)(g) = \int_N \chi(n)^{-1} f(ng)\, dn.$$

Where $d\eta(n) = dn$ is the usual Lebesgue measure on N. Then

$$\begin{aligned} s_{R_{g_1}f}(\chi)(g) &= \int_N \chi(n)^{-1} R_{g_1} f(ng)\, dn = \int_N \chi(n)^{-1} f(ngg_1)\, dn \\ &= s_f(\chi)(gg_1) = (R_{g_1} s_f(\chi))(g) \end{aligned}$$

and

$$\begin{aligned} s_{L_p f}(\chi)(g) &= \int_N \chi(n)^{-1} L_p f(ng)\, dn = \int_N \chi(n)^{-1} f(p^{-1}npp^{-1}g)\, dn \\ &= \int_N \chi(pnp^{-1})^{-1} \delta(p) f(np^{-1}g)\, dn \\ &= \int_N (p^{-1}\chi)(n)^{-1} \delta(p) f(np^{-1}g)\, dn \\ &= \delta(p) s_f(p^{-1}\chi)(p^{-1}g) = [\delta(p) L_p s_f(p^{-1}\chi)](g). \end{aligned}$$

This means that the action of $P \times G$ on $L^2(\hat{N}, E, \eta)$ is given by a vector bundle action, and hence, if Ω is the set of open orbits for the action of M on N, then

$$\begin{aligned} L^2(G) &\cong L^2(\hat{N}, E, \eta) \cong \bigoplus_{\chi \in \Omega} \operatorname{Ind}_{M_\chi N \times G}^{P \times G} (\operatorname{Ind}_N^P End(\chi)) \\ &\cong \bigoplus_{\chi \in \Omega} \operatorname{Ind}_{M_\chi N \times G}^{P \times G} \int_{\hat{M}_\chi} \tau^* \chi^* \otimes \operatorname{Ind}_{M_\chi N}^G \tau\chi\, d\nu(\tau) \\ &\cong \bigoplus_{\chi \in \Omega} \int_{\hat{M}_\chi} \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes L^2(M_\chi N \setminus G; \tau\chi)\, d\nu(\tau) \\ &\cong \bigoplus_{\chi \in \Omega} \int_{\hat{M}_\chi} \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes \int_{\hat{G}} W_{\chi,\tau}(\pi) \otimes \pi\, d\nu(\tau)\, d\tilde{\mu}(\pi) \\ &\cong \bigoplus_{\chi \in \Omega} \int_{\hat{G}} \int_{\hat{M}_\chi} W_{\chi,\tau}(\pi) \otimes \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes \pi\, d\nu(\tau)\, d\tilde{\mu}(\pi). \qquad (11) \end{aligned}$$

On the other hand, the Harish-Chandra Plancherel theorem says that

$$L^2(G) \cong \int_{\hat{G}} \pi^*|_P \otimes \pi\, d\mu(\pi). \qquad (12)$$

Hence from (11) and (12) $\tilde{\mu}$ is in the measure class of μ and

$$\pi^*|_P \cong \bigoplus_{\chi \in \Omega} \int_{\hat{M}_\chi} W_{\chi,\tau}(\pi) \otimes \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^*\, d\nu(\tau). \qquad (13)$$

Also observe that as an $M_\chi N \times G$-module

$$
\begin{aligned}
L^2(N \setminus G; \chi) &\cong \operatorname{Ind}_{M_\chi N \times M_\chi N}^{M_\chi N \times G} \operatorname{Ind}_{N \times N}^{M_\chi N \times M_\chi N} \chi^* \otimes \chi \\
&\cong \operatorname{Ind}_{M_\chi N \times M_\chi N}^{M_\chi N \times G} \int_{\hat{M}_\chi} \tau^* \chi * \otimes \tau \otimes \chi \, d\nu(\tau) \\
&\cong \int_{\hat{M}_\chi} \tau^* \chi * \otimes \operatorname{Ind}_{M_\chi N}^{G} \tau \chi \, d\nu(\tau) \\
&\cong \int_{\hat{M}_\chi} \tau^* \chi * \otimes \int_{\hat{G}} W_{\chi,\tau}(\pi) \otimes \pi \, d\nu(\tau) \, d\mu(\pi) \\
&\cong \int_{\hat{G}} \int_{\hat{M}_\chi} W_{\chi,\tau}(\pi) \otimes \tau^* \chi * \otimes \pi \, d\nu(\tau) \, d\mu(\pi) \qquad (14)
\end{aligned}
$$

Returning to the case $G = Sp(m, \mathbb{R})$, equation (13) reads

$$
\pi^*|_P \cong \bigoplus_{r+s=m} \int_{O(\hat{r,s})} W_{\chi_{r,s},\tau}(\pi) \otimes \operatorname{Ind}_{O(r,s)N}^{P} \tau^* \chi_{r,s}^* \, d\nu(\tau). \qquad (15)
$$

From this and equation (10)

$$
\begin{aligned}
L^2(Hom(\mathbb{R}^m, \mathbb{R}^n)) &\cong \int_{Sp(\hat{m,\mathbb{R}})} \pi^*|_P \otimes \Theta(\pi^*) \, d\mu(\pi) \\
&\cong \int_{Sp(\hat{m,\mathbb{R}})} \bigoplus_{r+s=m} \int_{O(\hat{r,s})} W_{\chi_{r,s},\tau}(\pi) \otimes \operatorname{Ind}_{O(r,s)N}^{P} \tau^* \chi_{r,s}^* \otimes \Theta(\pi^*) \, d\nu(\tau) \, d\mu(\pi) \\
&\cong \bigoplus_{r+s=m} \int_{Sp(\hat{m,\mathbb{R}})} \int_{O(\hat{r,s})} W_{\chi_{r,s},\tau}(\pi) \otimes \operatorname{Ind}_{O(r,s)N}^{P} \tau^* \chi_{r,s}^* \otimes \Theta(\pi^*) \, d\nu(\tau) \, d\mu(\pi).
\end{aligned}
$$

Now from above equation and equation (9)

$$
\begin{aligned}
&\bigoplus_{r+s=m} \int_{O(\hat{p,q})} \int_{O(\hat{r,s})} M_{\tau,r,s}(\pi) \otimes \operatorname{Ind}_{O(r,s)N}^{P} \tau^* \chi_{r,s}^* \otimes \pi \, d\nu(\tau) \, d\eta(\pi) \cong \\
&\qquad \bigoplus_{r+s=m} \int_{Sp(\hat{m,\mathbb{R}})} \int_{O(\hat{r,s})} W_{\chi_{r,s},\tau}(\pi) \otimes \operatorname{Ind}_{O(r,s)N}^{P} \tau^* \chi_{r,s}^* \otimes \Theta(\pi^*) \, d\nu(\tau) \, d\mu(\pi).
\end{aligned}
$$

Hence

$$
W_{\chi_{r,s},\tau}(\pi) \cong M_{\tau,r,s}(\Theta(\pi^*)) \qquad (16)
$$

and η is the pullback of μ under the map

$$
\Theta : Sp(\hat{m,\mathbb{R}}) \longrightarrow O(\hat{p,q}).
$$

Putting everything together we obtain the following interesting formulas:

$$
\begin{aligned}
\operatorname{Ind}_{O(p-r,q-s)}^{O(p,q)} 1 &\cong \int_{O(\hat{p,q})} \int_{O(\hat{r,s})} \tau^* \otimes M_{\tau,r,s}(\pi) \otimes \pi \, d\nu(\tau) \, d\tilde{\eta}(\pi) \\
&\cong \int_{Sp(\hat{m,\mathbb{R}})} \int_{O(\hat{r,s})} W_{\chi_{r,s},\tau}(\pi) \otimes \tau^* \otimes \Theta(\pi^*) \, d\nu(\tau) \, d\mu(\pi).
\end{aligned}
$$

and

$$
L^2(N \setminus Sp(m, \mathbb{R}); \chi_{r,s}) \cong \int_{Sp(\hat{m,\mathbb{R}})} \int_{O(\hat{r,s})} W_{\chi_{r,s},\tau}(\pi) \otimes \tau^* \otimes \pi \, d\nu(\tau) \, d\mu(\pi). \qquad (17)
$$

And also the equations

$$L^2(Hom(\mathbb{R}^m,\mathbb{R}^n)) \cong \bigoplus_{r+s=m} \int_{\hat{Sp(m,\mathbb{R})}} \int_{\hat{O(r,s)}} W_{\chi_{r,s},\tau}(\pi) \otimes \mathrm{Ind}_{M_\chi N}^{P}\, \tau^* \chi_{r,s}^* \otimes \Theta(\pi^*)\, d\nu(\tau)\, d\mu(\pi)$$

and

$$L^2(Sp(m,\mathbb{R})) \cong \bigoplus_{r+s=m} \int_{\hat{Sp(m,\mathbb{R})}} \int_{\hat{O(r,s)}} W_{\chi_{r,s},\tau}(\pi) \otimes \mathrm{Ind}_{M_\chi N}^{P}\, \tau^* \chi_{r,s}^* \otimes \pi\, d\nu(\tau)\, d\mu(\pi).$$

We are thus left with the calculation of $W_{\chi_{r,s},\tau}$. We conjecture that

$$W_{\chi_{r,s},\tau}(\pi) \cong Wh_{\chi_{r,s},\tau}(\pi)$$

where

$$Wh_{\chi_{r,s},\tau}(\pi) := \left\{ \lambda : \pi^\infty \longrightarrow \tau^\infty \;\middle|\; \begin{array}{c} \lambda(\pi(mn)v) = \chi_{r,s}(n)\tau(m)\lambda(v) \\ \text{for all } m \in M_{\chi_{r,s}} \cong O(r,s),\, n \in N. \end{array} \right\}$$