Representations of Parabolic Groups

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Abstract

Some results on representations of parabolic groups. This is only a reference file.

1 Irreducible representations of Parabolic Groups

Theorem 1 *Let P be a parabolic subgroup, and let $P = MN$ be its Langlands decomposition. If (π, H) is an irreducible unitary representation of P, then*

$$H \cong \mathrm{Ind}_{M_\chi N}^{P} \tau\chi \qquad \textit{with } \tau \in \hat{M}_\chi,\ \chi \in \hat{N}.$$

Proof. As an N-module, we have that

$$H \cong \int_{\hat{N}} E_\chi \, d_\nu(\chi),$$

where $E_\chi \cong L_\chi \otimes V_\chi$, $V_\chi \in \hat{N}$, and L_χ is a multiplicity space. This means that there exists a vector bundle

$$\begin{array}{c} E \\ \downarrow \\ \hat{N} \end{array}$$

and a measure ν on $\hat{N}$, such that

$$H \cong L^2(\hat{N}, E, \nu) := \{s : \hat{N} \to E \,|\, s(\chi) \in E_\chi, \int_{\hat{N}} \|s(\chi)\|^2 \, d_\nu(\chi) < \infty\}$$

under the action

$$(\pi(n) \cdot s)(\chi) = \chi(n)s(\chi).$$

Under this isomorphism we can extend this action of N on $L^2(\hat{N}, E, \nu)$ to an action of P on the same space.

Let $m \in M$, and define

$$\begin{array}{c} E^m \\ \downarrow \\ \hat{N} \end{array}$$

to be the vector bundle such that $E^m_\chi = E_{m \cdot \chi}$. Define a measure ν_m on $\hat{N}$ by

$$\nu_m(X) = \nu(m \cdot X) \qquad \text{for } X \subset \hat{N} \text{ a measurable set}$$

and define

$$\tau(m) : L^2(\hat{N}, E, \nu) \longrightarrow L^2(\hat{N}, E^m, \nu_m)$$

by

$$(\tau(m)s)(\chi) = (\phi(m)s)(m \cdot \chi).$$

We claim that $\tau(m)$ is an isometry. Effectively

$$\begin{aligned} \|\tau(m)s\|_m^2 &= \int_{\hat{N}} \|(\tau(m)s)(\chi)\|^2 \, d\nu_m(\chi) \\ &= \int_{\hat{N}} \|(\pi(m)s)(m \cdot \chi)\|^2 \, d_\nu(m \cdot \chi) \\ &= \int_{\hat{N}} \|(\pi(m)s)(\chi)\|^2 \, d_\nu(\chi) \\ &= \|\pi(m)s)\|^2 = \|s\|^2. \end{aligned}$$

where the last equality comes from the fact that the action of P is unitary. Now if we define an action of N on $L^2(\hat{N}, E^m, \nu_m)$ by

$$(\pi_m \cdot s)(\chi) = \chi(n)s(\chi),$$

then $\tau(m)$ becomes an N-intertwiner. Effectively,

$$\begin{aligned} \tau(m)(\pi(n)s)(\chi) &= \pi(m)\pi(n)s(m \cdot \chi) \\ &= \pi(mnm^{-1})(\pi(m)s)(m \cdot \chi) \\ &= (m \cdot \chi)(mnm^{-1})(\pi(m)s)(m \cdot \chi) \\ &= \chi(m^{-1}mnm^{-1}m)(\pi(m)s)(m \cdot \chi) \\ &= \chi(n)(\tau(m)s)(\chi) = (\pi_m(n)\tau(m)s)(\chi). \end{aligned}$$

But now since N is a CCR group the N-interwiner

$$\tau(m) : L^2(\hat{N}, E, \nu) \longrightarrow L^2(\hat{N}, E^m, \nu_m)$$

should come from a morphism of vector bundles

$$\tilde{\tau}(m) : E \longrightarrow E^m,$$

that is, $(\tau(m)s)(\chi) = \tilde{\tau}(m)s(\chi)$, and hence

$$\begin{aligned} (\tau(m)s)(\chi) &= \tilde{\tau}(m)s(\chi) \\ (\pi(m)s)(m \cdot \chi) &= \tilde{\tau}(m)s(\chi) \end{aligned}$$

which says that

$$(\pi(m)s)(\chi) = \tilde{\tau}(m)s(m^{-1} \cdot \chi).$$

Now since $L^2(\hat{N}, E, \nu)$ is irreducible as a representation of P, the support of ν should be contained in a unique M-orbit on $\hat{N}$, and hence

$$L^2(\hat{N}, E, \nu) \cong L^2(M/M_\chi, E) \cong \operatorname{Ind}_{M_\chi N}^P E_\chi.$$

Using again that $L^2(\hat{N}, E, \nu)$ is irreducible we conclude that $E_\chi \cong \tau\chi$ with $\tau \in \hat{M}_\chi$, $\chi \in \hat{N}$. Putting all of this together we get that

$$H \cong \operatorname{Ind}_{M_\chi N}^P \tau\chi$$

as we wanted to show. ■

2 Decomposition of $L^2(P, d_r p)$ under the action of $P \times P$

We will now decompose $L^2(P, d_r p)$ under the action of $P \times P$ given by

$$(p_1, p_2) \cdot f = \delta(p_1)^{-1} L_{p_1} R_{p_2} f.$$

As a left N-module

$$\begin{aligned} L^2(P) &\cong \operatorname{Ind}_N^P \operatorname{Ind}_1^N 1 \cong \operatorname{Ind}_N^P(L^2(N)) \\ &\cong \operatorname{Ind}_N^P\left(\int_{\hat{N}} HS(V_\chi)\, d\mu(\chi)\right) \\ &\cong \int_{\hat{N}} \operatorname{Ind}_N^P HS(V_\chi)\, d\mu(\chi) \cong L^2(\hat{N}, E, \mu), \end{aligned}$$

with $E_\chi = HS(V_\chi)$. The isomorphism is given in the following way: Given $f \in L^2(P)$, define $s_f \in L^2(\hat{N}, E, \nu)$ by

$$s_f(\chi)(p) = \int_N \chi(n)^{-1} f(np)\, dn.$$

Then

$$\begin{aligned} s_{R_{p_1} f}(\chi)(p) &= \int_N \chi(n)^{-1} R_{p_1} f(np)\, dn = \int_N \chi(n)^{-1} f(npp_1)\, dn \\ &= s_f(\chi)(pp_1) = (R_{p_1} s_f(\chi))(p), \end{aligned}$$

and

$$\begin{aligned} s_{L_{p_1} f}(\chi)(p) &= \int_N \chi(n)^{-1} \delta(p_1)^{-1} L_{p_1} f(np)\, dn \\ &= \int_N \chi(n)^{-1} \delta(p_1)^{-1} f(p_1^{-1} n p_1 p_1^{-1} p)\, dn \\ &= \int_N \chi(p_1 n p_1^{-1})^{-1} f(n p_1^{-1} p)\, dn \\ &= \int_N (p_1^{-1}\chi)(n)^{-1} f(n p_1^{-1} p)\, dn \\ &= s_f(p_1^{-1}\chi)(p_1^{-1} p) = [L_{p_1} s_f(p_1^{-1}\chi)](p). \end{aligned}$$

Therefore

$$\begin{aligned} L^2(\hat{N}, E, \mu) &\cong L^2(\hat{N}/M, E, \tilde{\mu}) \\ &\cong \int_{\hat{N}/M} \operatorname{Ind}_{M_\chi N \times P}^{P \times P} \operatorname{Ind}_N^P HS(V_\chi)\, d\tilde{\mu}(\chi) \\ &\cong \int_{\hat{N}/M} \operatorname{Ind}_{M_\chi N}^P\left(\int_{\hat{M}_\chi} \tau^* \chi^* \otimes \operatorname{Ind}_{M_\chi N}^P \tau\chi\right) d\nu(\tau)\, d\tilde{\mu}(\chi) \\ &\cong \int_{\hat{N}/M} \int_{\hat{M}_\chi} \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes \operatorname{Ind}_{M_\chi N}^P \tau\chi\, d\nu(\tau)\, d\tilde{\mu}(\chi). \end{aligned}$$

3 Decomposition of $L^2(G)$ under the action of $P \times G$

We will now consider $L^2(G)$ as a $P \times G$ module. Reasoning as in the $L^2(P)$ case we have an isomorphism

$$L^2(G) = L^2(\hat{N}, E, \mu)$$

with $E_\chi = \operatorname{Ind}_N^P HS(V_\chi)$ given in the follwing way: given $f \in L^2(G)$, define $s_f \in L^2(\hat{N}, E, \mu)$ by

$$s_f(\chi)(g) = \int_N \chi(n)^{-1} f(ng)\, dn.$$

Then

$$\begin{aligned} s_{R_{g_1}f}(\chi)(g) &= \int_N \chi(n)^{-1} R_{g_1} f(ng)\, dn = \int_N \chi(n)^{-1} f(ngg_1)\, dn \\ &= s_f(\chi)(gg_1) = (R_{g_1} s_f(\chi))(g) \end{aligned}$$

and

$$\begin{aligned} s_{L_p f}(\chi)(g) &= \int_N \chi(n)^{-1} L_p f(ng)\, dn = \int_N \chi(n)^{-1} f(p^{-1}npp^{-1}g)\, dn \\ &= \int_N \chi(pnp^{-1})^{-1} \delta(p) f(np^{-1}g)\, dn \\ &= \int_N (p^{-1}\chi)(n)^{-1} \delta(p) f(np^{-1}g)\, dn \\ &= \delta(p) s_f(p^{-1}\chi)(p^{-1}g) = [\delta(p) L_p s_f(p^{-1}\chi)](g). \end{aligned}$$

Therefore

$$\begin{aligned} L^2(\hat{N}, E, \mu) &\cong L^2(\hat{N}/M, E, \tilde{\mu}) \\ &\cong \int_{\hat{N}/M} \operatorname{Ind}_{M_\chi N \times G}^{P \times G} \operatorname{Ind}_N^P HS(V_\chi)\, d\tilde{\mu}(\chi) \\ &\cong \int_{\hat{N}/M} \operatorname{Ind}_{M_\chi N \times G}^{P \times G} (\int_{\hat{M}_\chi} \tau^* \chi^* \otimes \operatorname{Ind}_{M_\chi N}^G \tau\chi)\, d\nu(\tau)\, d\tilde{\mu}(\chi) \\ &\cong \int_{\hat{N}/M} \int_{\hat{M}_\chi} \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes L^2(M_\chi N \backslash G; \tau\chi)\, d\nu(\tau)\, d\tilde{\mu}(\chi). \\ &\cong \int_{\hat{N}/M} \int_{\hat{M}_\chi} \int_{\hat{G}} \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes W_{\chi,\tau}(\pi) \otimes \pi\, d\nu(\tau)\, d\tilde{\mu}(\chi)\, d\tilde{\eta}(\pi). \end{aligned}$$

On the other hand

$$L^2(G) \cong \int_{\hat{G}} \pi^*|_P \otimes \pi\, d\eta(\pi).$$

Hence $\tilde{\eta}$ is in the measure class of the Plancherel measure, and

$$\pi^*|_P \cong \int_{\hat{N}/M} \int_{\hat{M}_\chi} \operatorname{Ind}_{M_\chi N}^P \tau^* \chi^* \otimes W_{\chi,\tau}(\pi)\, d\nu(\tau)\, d\tilde{\mu}(\chi).$$