A CONJECTURE OF SAKELLARIDIS-VENKATESH ON THE UNITARY SPECTRUM OF SPHERICAL VARIETIES.

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to Nolan Wallach,
with admiration and appreciation

1. INTRODUCTION

The spectral decomposition of the unitary representation $L^2(H\backslash G)$ when $X = H\backslash G$ is a symmetric space has been studied extensively, especially in the case when G is a real Lie group. In particular, through the work of many authors (such as [7], [20], [26], [3] and [1]), one now has the full Plancherel theorem in this setting.

In a recent preprint [23], Sakellaridis and Venkatesh considered the more general setting where $X = H\backslash G$ is a spherical variety and G is a real or p-adic group. Motivated by the study of periods in the theory of automorphic forms and the comparison of relative trace formulas, they formulated an approach to this problem in the framework of Langlands functoriality. More precisely, led by and refining the work of Gaitsgory-Nadler [9] in the geometric Langlands program, they associated to a spherical variety $X = H\backslash G$ (satisfying some additional technical hypotheses)

- a dual group $\check{G}_X$;
- a natural map $\iota : \check{G}_X \times SL_2(\mathbb{C}) \longrightarrow \check{G}$

The map ι induces a map from the set of tempered L-parameters of G_X to the set of Arthur parameters of G, and if one is very optimistic, it may even give rise to a map

$$\iota_* : \widehat{G}_X \longrightarrow \widehat{G}$$

where G_X is a (split) group with dual group $\check{G}_X$ and $\widehat{G}_X$ and $\widehat{G}$ refer to the unitary dual of the relevant groups. Assuming for simplicity that this is the case, one has the following conjecture:

Sakellaridis-Venkatesh Conjecture

One has a spectral decomposition

$$L^2(H\backslash G) \cong \int_{\widehat{G}_X} W(\pi) \otimes \iota_*(\pi) \, d\mu(\pi)$$

where μ is the Plancherel measure of $\widehat{G}_X$ and $W(\pi)$ is some (finite-dimensional) multiplicity space. The multiplicity space $W(\pi)$ should be related to the space of continuous H-invariant functionals on the representation $\iota_*(\pi)$.

In particular, the class of the spectral measure of $L^2(H\backslash G)$ is absolutely continuous with respect to that of the pushforward by ι_* of the Plancherel measure on $\widehat{G}_X$, and its support is contained in the set of those Arthur parameters of G which factor through ι. Moreover, from the point of view of Arthur parameters, the multiplicity space should be related to the number of inequivalent ways an Arthur parameter valued in $\check{G}$ can be lifted to $\check{G}_X$ (i.e. factored through ι).

The main purpose of this paper is to verify the above conjecture in many cases when $H\backslash G$, or equivalently G_X, has low rank, and to specify the multiplicity space $W(\pi)$. In particular, we

demonstrate this conjecture for many cases when G_X has rank 1, and also some cases when G_X has rank 2 or 3 (see the tables in [23, §15 and §16]). More precisely, our main result is:

Theorem 1. *The conjecture of Sakellaridis-Venkatesh holds for the spherical varieties $X = H\backslash G$ listed in the following tables.*

X	$GL_{n-1}\backslash GL_n$	$SO_{n-1}\backslash SO_n$	$Sp_{2n-2}\backslash Sp_{2n}$
G_X	GL_2	$\widetilde{SL_2}$	$SO(4)$

TABLE 1. Classical cases

X	$SO_3\backslash SL_3$	$Sp_6\backslash SL_6$	$SL_3\backslash G_2$	$(J,\psi)\ \backslash G_2$	$SU_3\backslash Spin_7$
G_X	$\widetilde{SL_3}$	SL_3	$\widetilde{SL_2}$	PGL_3	$(Spin_3 \times Spin_5)/\Delta\mu_2$

X	$G_2\backslash Spin_7$	$G_2\backslash Spin_8$	$Spin_9\backslash F_4$	$F_4\backslash E_6$	
G_X	SL_2	$SL_2^3/\Delta\mu_2$	PGL_2	SL_3	

TABLE 2. Exceptional cases

For the classical cases, the precise results are contained in Theorem 6 in §3.8 and the ensuing discussion in §3.9. We note that Theorem 6 gives the spectral decomposition, in the spirit of the Sakellaridis-Venkatesh conjecture, of the so-called generalized Stiefel manifolds, which are homogeneous but not necessarily spherical varieties. The exceptional cases are covered in §8.4 (Theorem 16), §8.5 and §9. We note that over $\mathbb{R}$, Kobayashi has given in [15] an explicit description of the discrete spectrum of the generalized Stiefel manifolds in terms of $A_{\mathfrak{q}}(\lambda)$ modules; he has also described in [16] the spectrum of certain special spherical varieties such as $SL_3\backslash G_2$ which can be related to symmetric spaces (as we explain in §4.4). His viewpoint is quite disjoint from that of this paper.

Theorem 1 is proved using the technique of theta correspondence. More precisely, it turns out that for the groups listed in the above table, one has a reductive dual pair

$$G_X \times G \subset S$$

for some larger group S. One then studies the restriction of the minimal representation of S to the subgroup $G_X \times G$. In the context of theta correspondence in smooth representation theory, one can typically show the following rough statement:

A representation π of G has ψ-generic (and hence nonzero) theta lift to G_X

$$\Updownarrow$$

π has nonzero H-period.

Our main theorem is thus the L^2-manifestation of this phenomenon, giving a description of $L^2(H\backslash G)$ in terms of $L^2(G_X)$.

This idea is not really new: a well known example of this kind of result is the correspondence between the irreducible components of the spherical harmonics on $\mathbb{R}^n$ under the action of $O(n,\mathbb{R})$, and holomorphic discrete series of the group $\widetilde{SL}(2,\mathbb{R})$, the double cover of $SL(2,\mathbb{R})$. Another example is given by the classical paper of Rallis and Schiffmann [21] where they used the oscillator representation to relate the discrete spectrum of $L^2(O(p,q-1)\backslash O(p,q))$ with the discrete series representations of $\widetilde{SL}(2,\mathbb{R})$. Later, Howe [12] showed how these results can be inferred from his general

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theory of reductive dual pairs, and essentially provided a description of the Plancherel measure of $L^2(O(p, q-1)\backslash O(p, q))$ in terms of the representation theory of $\widetilde{SL}(2, \mathbb{R})$. Then Ørsted and Zhang [30] proved a similar result for the space $L^2(U(p, q-1)\backslash U(p, q))$ in terms of the representation theory of $U(1, 1)$. We give a more steamlined treatment of these classical cases in Section 2, which accounts for Table 1. The rest of the paper is then devoted to the exceptional cases listed in Table 2.

2. **Classical Dual Pairs**

We begin by introducing the classical dual pairs.

2.1. **Division algebra D.** Let $\mathfrak{k}$ be a local field, and let $|\cdot|$ denote its absolute value. Let $D = \mathfrak{k}$, a quadratic field extension of $\mathfrak{k}$ or the quaternion division $\mathfrak{k}$-algebra, and let $x \mapsto \overline{x}$ be its canonical involution. The case when D is the split quadratic algebra or quaternion algebra can also be included in the discussion, but for simplicity, we shall stick with division algebras. We have the *reduced trace* map $Tr : D \to \mathfrak{k}$ and the *reduced norm* map $Q : D \to \mathfrak{k}$. If $D \neq \mathfrak{k}$, one has $Tr(x) = x + \overline{x} \in \mathfrak{k}$ and $Q(x) = x \cdot \overline{x} \in \mathfrak{k}$.

2.2. **Hermitian D-modules.** Let V and W be two right D-modules. We will denote the set of right D-module morphisms between V and W by

$$Hom_D(V, W) = \{T : V \longrightarrow W \,|\, T(v_1 a + v_2 b) = T(v_1)a + T(v_2)b \text{ for all } v_1, v_2 \in V, a, b \in D\}.$$

In the same way, if V and W are two left D-modules, we set

$$Hom_D(V, W) = \{T : V \longrightarrow W \,|\, (av_1 + bv_2)T = a(v_1)T + b(v_2)T \text{ for all } v_1, v_2 \in V, a, b \in D\}.$$

If $V = W$, we will denote this set by $End_D(V)$. Notice that for right D-module morphisms we are putting the argument on the right, while for left D-module morphisms we are putting it on the left.

In general, for every statement involving *right* D-modules one can make an analogous one involving *left* D-modules. From now on, we will focus on right D-modules, and we will let the reader with the task of making the corresponding definitions and statements involving left D-modules. Set

$$GL(V, D) = \{T \in End_D(V) \,|\, \text{T is invertible}\}.$$

When it is clear from the context what the division algebra is, we will just denote this group by $GL(V)$.

Let V' be the set of right D-linear functionals on V. There is a natural left D-module structure on V' given by setting

$$(a\lambda)(v) = a\lambda(v), \quad \text{for all } a \in D, v \in V, \text{ and } \lambda \in V'.$$

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Observe that with this structure, $W \otimes_D V'$ is naturally isomorphic to $Hom_D(V, W)$ as a $\hat{k}$-vector space. Given $T \in Hom_D(V, W)$, we will define an element in $Hom_D(W', V')$, which we will also denote T, by setting $(\lambda T)(v) := \lambda(Tv)$. This correspondence gives rise to natural isomorphisms between $End_D(V)$ and $End_D(V')$ and between $GL(V)$ and $GL(V')$.

Definition 2. *Let $\varepsilon = \pm 1$. We say that (V, B) is a right ε-Hermitian D-module, if V is a right D-module and B is an ε-Hermitian form, i.e $B : V \times V \longrightarrow D$ is a map such that*

(1) B is sesquilinear. *That is, for all v_1, v_2, $v_3 \in V$, a, $b \in D$,*

$$B(v_1, v_2 a + v_3 b) = B(v_1, v_2)a + B(v_1, v_3)b \quad and \quad B(v_1 a + v_2 b, v_3) = \overline{a}B(v_1, v_3) + \overline{b}B(v_2, v_3).$$

(2) B is ε-Hermitian. That is,

$$B(v, w) = \varepsilon\overline{B(w, v)} \qquad for\ all\ v, w \in V.$$

(3) B is non-degenerate.

Usually, 1-Hermitian D-modules are simply called Hermitian, while -1-Hermitian D-modules are called skew-Hermitian. To define left ε-Hermitian D-modules (V, B), we just have to replace the sesquilinear condition by

$$B(av_1 + bv_2, v_3) = aB(v_1, v_3) + bB(v_2, v_3) \quad and \quad B(v_1, av_2 + bv_3) = B(v_1, v_2)\overline{a} + B(v_1, v_3)\overline{b},$$

for all v_1, v_2, $v_3 \in V$, a, $b \in D$.

Given a right ε-Hermitian D-module (V, B), we will define

$$G(V, B) = \{g \in GL(V) \,|\, B(gv, gw) = B(v, w) \text{ for all } v, w \in V\},$$

to be the subgroup of $GL(V)$ preserving the ε-Hermitian form B. When there is no risk of confusion regarding B, we will denote this group just by $G(V)$. Later on, we shall sometimes need use the same notation to denote a covering group of $G(V, B)$; see §2.4.

Given a right ε-Hermitian D-module (V, B), we can construct a left ε-Hermitian D-module (V^*, B^*) in the following way: as a set, V^* will be the set of symbols $\{v^* \,|\, v \in V\}$. Then we give V^* a left D-module structure by setting, for all v, $w \in V$, $a \in D$,

$$v^* + w^* = (v + w)^* \text{ and } av^* = (v\overline{a})^*.$$

Finally, we set

$$B^*(v^*, w^*) = \overline{B(w, v)} \qquad \text{for all } v, w \in V.$$

In an analogous way, if V is a left D-module, we can define a right D-module V^*, and V^{**} is naturally isomorphic with V. Given $T \in End_D(V)$, we can define $T^* \in End_D(V^*)$ by setting $v^*T^* := (Tv)^*$. With this definition, it is easily seen that $(TS)^* = S^*T^*$, for all S, $T \in End_D(V)$. Therefore the map $g \mapsto (g^*)^{-1}$ defines an algebraic group isomorphism between $GL(V)$ and $GL(V^*)$.

Now observe that the form B induces a left D-module isomorphism $B^\flat : V^* \longrightarrow V'$ given by $B^\flat(v^*)(w) = B(v, w)$ for v, $w \in V$. In what follows, we will make implicit use of this map to identify this two spaces. With this identification we can think of T^* as a map in $End_D(V)$ defined by $v^*(T^*w) := (v^*T^*)(w)$, i.e, T^* is defined by the condition that

$$B(v, T^*w) = B(Tv, w) \qquad \text{for all } v, w \in V.$$

Observe that this agrees with the usual definition of T^*.

A D-submodule $X \subset V$ is said to be *totally isotropic* if $B|_{X \times X} = 0$. If X is a totally isotropic submodule, then there exists a totally isotropic submodule $Y \subset V$ such that $B|_{X \oplus Y \times X \oplus Y}$ is non-degenerate. If we set

$$U = (X \oplus Y)^\perp := \{u \in V \,|\, B(u, w) = 0 \text{ for all } w \in X \oplus Y\},$$

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then $V = X \oplus Y \oplus U$, and $B|_{U \times U}$ is non-degenerate. In this case we say that X and Y are totally isotropic, *complementary* submodules. Observe that then $B^\flat|_{Y^*} : Y^* \longrightarrow X'$ is an isomorphism. As before we will make implicit use of this isomorphism to identify Y^* with X'.

2.3. **Reductive dual pairs.** Let (V, B_V) be a right ε_V-Hermitian D-module and (W, B_W) a right ε_W-Hermitian D-module such that $\varepsilon_V \varepsilon_W = -1$. On the $\Bbbk$-vector space $V \otimes_D W^*$ we can define a symplectic form B by setting

$$B(v_1 \otimes_D \lambda_1, v_2 \otimes_D \lambda_2) = \text{Tr}(B_W(w_1, w_2) B_V^*(\lambda_2, \lambda_1)) \qquad \text{for all } v_1, v_2 \in V \text{ and } \lambda_1, \lambda_2 \in V^*.$$

Let

$$Sp(V \otimes_D W^*) = \{g \in GL(V \otimes_D W^*, \Bbbk) \,|\, B(gv, gw) = B(v, w) \text{ for all } v, w \in V \otimes_D W^*\}.$$

Observe that

$$Sp(V \otimes_D W^*) = G(V \otimes_D W^*, B) = G(V \otimes_D W^*).$$

Moreover, there is a natural map $G(V) \times G(W) \longrightarrow Sp(V \otimes_D W^*)$ given by

$$(g_1, g_2) \cdot v \otimes_D \lambda = g_1 v \otimes \lambda g_2^*.$$

We will use this map to identify $G(V)$ and $G(W)$ with subgroups of $Sp(V \otimes_D W^*)$. These two subgroups are mutual commutants of each other, and is an example of a *reductive dual pair*.

2.4. **Metaplectic cover.** The group $Sp(V \otimes_D W^*)$ has an S^1- cover $Mp(V \otimes_D W^*)$ which is called a metaplectic group. It is known that this S^1-cover splits over the subgroups $G(V)$ and $G(W)$, except when V is an odd dimensional quadratic space, in which it does not split over $G(W)$. In this exceptional case, we shall simply redefine $G(W)$ to be the induced double cover, so as to simplify notation. We remark also that though the splittings (when they exist) are not necessarily unique, the precise choice of the splittings is of secondary importance in this paper.

2.5. **Siegel parabolic.** Assume in addition that there is a complete polarization $W = E \oplus F$, where E, F, are complementary totally isotropic subspaces of W. We will use the ε_W-Hermitian form B_W to identify F^* with E' by setting $f^*(e) = B_W(f, e)$. Observe that this identification induces an identification between E^* and F' given by

$$e^*(f) = \overline{f^*(e)} = \overline{B_W(f, e)} = \varepsilon_W B_W(e, f).$$

In what follows, we will use this identifications between F^* and E', and between E^* and F'.

Let

$$P = \{p \in G(W) \,|\, p \cdot E = E\}$$

be the Siegel parabolic subgroup of $G(W)$, and let $P = MN$ be its Langlands decomposition. To give a description of the groups M and N, we introduce some more notation.

Let $A \in End_D(E)$. We will define $A^* \in End_D(F)$, by setting, for all $e \in E$, $f \in F$,

$$(1) \qquad\qquad\qquad\qquad B_W(e, A^*f) = B_W(Ae, f).$$

Now given $T \in Hom_D(F, E)$, define $T^* \in Hom_D(F, E)$ by setting, for all $f_1, f_2 \in F$,

$$(2) \qquad\qquad\qquad\qquad B_W(f_1, T^*f_2) = \varepsilon_W B_W(Tf_1, f_2).$$

Given $\varepsilon = \pm 1$, set

$$Hom_D(F, E)_\varepsilon = \{T \in Hom_D(F, E) \,|\, T^* = \varepsilon T\}.$$

It is then clear that $Hom_D(F, E) = Hom_D(F, E)_1 \oplus Hom_D(F, E)_{-1}$.

Now we have:

$$M = \left\{ \begin{bmatrix} A & \\ & (A^*)^{-1} \end{bmatrix} \,\middle|\, A \in GL(E) \right\} \cong GL(E)$$

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and

$$N = \left\{ \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \,\middle|\, X^* = -\varepsilon_W X \right\} \cong Hom_D(F, E)_{-\varepsilon_W}.$$

2.6. **Characters of N.** Given a nontrivial character $\chi : \mathfrak{k} \to \mathbb{C}^\times$ and $Y \in Hom_D(E, F)_{-\varepsilon_W}$, define a character

$$\chi_Y \left(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \right) = \chi(\mathrm{Tr}_F(YX)).$$

Here Tr_F is the trace of $YX : F \longrightarrow F$ seen as a map between $\mathfrak{k}$-vector spaces. The map $Y \mapsto \chi_Y$ defines a group isomorphism between $Hom_D(E, F)_{-\varepsilon_W}$ and $\hat{N}$.

Observe that the adjoint action of M on N induces an action of M on $\hat{N}$. Using the isomorphisms of $M \cong GL(E)$ and $\hat{N} \cong Hom_D(E, F)_{-\varepsilon_W}$, we can describe the action of M on $\hat{N}$ by the formula

$$A \cdot Y = (A^*)^{-1} Y A^{-1} \quad \text{for all } A \in GL(E),\ Y \in Hom_D(E, F)_{-\varepsilon_W}.$$

Given $Y \in Hom_D(E, F)_{-\varepsilon_W}$ we can define a $-\epsilon_W$-Hermitian form on E, that we will also denote Y, by setting

$$Y(e_1, e_2) = e_1^*(Ye_2) = \varepsilon_W B_W(e_1, Ye_2).$$

Hence the action of M on $\hat{N}$ is equivalent to the action of $GL(E)$ on sesquilinear, $-\varepsilon_W$-Hermitian forms on E.

Let Ω be the set of orbits for the action of M on $\hat{N}$. Given $Y \in Hom_D(E, F)_{-\varepsilon_W}$, let $\mathcal{O} = \mathcal{O}_Y$ be its orbit under the action of $GL(E)$ and set

$$M_{\chi_Y} = \{m \in M \,|\, \chi_Y(m^{-1}nm) = \chi_Y(n) \text{ for all } n \in N\}.$$

Using the identification of M with $GL(E)$, and of $\hat{N}$ with $Hom_D(E, F)_{-\varepsilon_W}$, we see that

$$M_{\chi_Y} \cong \{A \in GL(E) \,|\, (A^*)^{-1}Y A^{-1} = Y\} = \{A \in GL(E) \,|\, Y = A^* Y A\}.$$

3. Oscillator Representation and Theta Correspondence

After the preparation of the previous section, we can now consider the theta correspondence associated to the dual pair $G(V) \times G(W)$ and use it to establish certain cases of the Sakellaridis-Venkatesh conjecture for classical groups.

3.1. **Oscillator representation.** Fix a nontrivial unitary character χ of $\mathfrak{k}$. Associated to this character, there exists a very special representation of the metaplectic group, called the oscillator representation Π of $Mp(V \otimes_D W^*)$. On restricting this representation to $G(V) \times G(W)$, one may write

$$(3) \qquad\qquad \Pi|_{G(W) \times G(V)} = \int_{G(W)^\wedge} \pi \otimes \Theta(\pi) \, d\mu_\theta(\pi),$$

as a $G(W) \times G(V)$-module, for some measure μ_θ on $G(W)^\wedge$ and where $\Theta(\pi)$ is a (possibly zero, possibly reducible) unitary representation of $G(V)$. We shall call the map Θ the L^2-theta correspondence.

3.2. **Smooth vs. L^2-theta correspondence.** One may consider the above restriction of the oscillator representation in the category of smooth representations (the so-called smooth theta correspondence). Namely, for $\pi \in G(W)^\wedge$, let π^∞ denote the subspace of smooth vectors of π. Then one may consider the maximal π^∞-isotypic quotient of Π^∞ (the smooth representation underlying Π), which has the form $\pi^\infty \otimes \Theta^\infty(\pi^\infty)$ for some smooth representation $\Theta^\infty(\pi^\infty)$ of $G(V)$, known as the (big) smooth theta lift of π^∞. It is known that $\Theta^\infty(\pi^\infty)$ is an admissble representation of finite length. Moreover, unless $\mathfrak{k}$ is a 2-adic field, one knows further that $\Theta^\infty(\pi^\infty)$ has a unique irreducible quotient $\theta^\infty(\pi^\infty)$ (the small smooth theta lift of π^∞); this is the so-called Howe duality conjecture. In any case, we may define $\theta^\infty(\pi^\infty)$ to be the maximal semisimple quotient of $\Theta^\infty(\pi^\infty)$.

It is natural to wonder how the L^2-theta correspondence and the smooth theta correspondence are related. One can show using the machinery developed in Bernstein's paper [2] that, in the context of (3), for μ_θ-almost all π, there is a nonzero surjective equivariant map

$$\Pi^\infty \longrightarrow \pi^\infty \otimes \Theta(\pi)^\infty.$$

Such a map necessarily factors through:

$$\Pi^\infty \twoheadrightarrow \pi^\infty \otimes \Theta^\infty(\pi^\infty) \twoheadrightarrow \pi^\infty \otimes \Theta(\pi)^\infty,$$

so that one has a surjection

$$\Theta^\infty(\pi^\infty) \twoheadrightarrow \Theta(\pi)^\infty.$$

Thus, we see that $\Theta(\pi)^\infty$ is of finite length and unitarizable, so that $\Theta(\pi)^\infty$ is semisimple. Hence, we have a surjection

$$\theta^\infty(\pi^\infty) \twoheadrightarrow \Theta(\pi)^\infty.$$

Since $\theta^\infty(\pi^\infty)$ is semisimple, we deduce that

$$\Theta(\pi)^\infty \subseteq \theta^\infty(\pi^\infty),$$

so that $\Theta(\pi)$ is a direct sum of finitely many irreducible unitary representations for μ_θ-almost all π. Indeed, if $\mathfrak{k}$ is not a 2-adic field, $\Theta(\pi)$ is irreducible with

$$\Theta(\pi)^\infty = \theta^\infty(\pi^\infty)$$

for μ_θ-almost all π.

Hence, the L^2-theta correspondence gives a map

$$\Theta : G(W)^\wedge \longrightarrow R_{\geq 0}(G(V)^\wedge)$$

where $R_{\geq 0}(G(V)^\wedge)$ is the Grothendieck semigroup of unitary representations of $G(V)$ of finite length. If $\mathfrak{k}$ is not 2-adic, Θ takes value in $G(V)^\wedge \cup \{0\}$. Moreover, one has the compatibility of L^2-theta lifts (considered in this paper) with the smooth theta lifts.

3.3. **Restriction to $P \times G(V)$.** We may restrict Π further to $P \times G(V)$. By Mackey theory, for a unitary representation π of $G(W)$,

$$(4) \qquad\qquad \pi|_P = \bigoplus_{\mathcal{O}_Y \in \Omega} \mathrm{Ind}_{M_{\chi_Y} N}^P W_{\chi_Y}(\pi),$$

where $W_{\chi_Y}(\pi)$ is an $M_{\chi_Y} N$-module such that $n \cdot \lambda = \chi_Y(n)\lambda$, for all $n \in N$, $\lambda \in W_{\chi_Y}(\pi)$.

Therefore, from (3) and (4), we have:

$$(5) \qquad\qquad \Pi = \bigoplus_{\mathcal{O}_Y \in \Omega} \int_{A \subset \widehat{G}(W)} \mathrm{Ind}_{M_{\chi_Y} N}^P W_{\chi_Y}(\pi) \otimes \Theta(\pi) \, d\mu_\theta(\pi).$$

3.4. **The Schrödinger model.** On the other hand, we may compute the restriction of Π to $P \times G(V)$ using an explicit model of Π. The complete polarization $W = E \oplus F$ induces a complete polarization

$$V \otimes_D W^* = V \otimes_D E^* \oplus V \otimes_D F^*.$$

With the identifications introduced above, $V \otimes_D F^* = Hom_D(E, V)$, and the oscillator representation Π can be realized on the Hilbert space $L^2(Hom_D(E, V))$; this realization of Π is called the Schrödinger model. The action of $P \times G(V)$ in this model can be described as follows.

Let $B_V^\flat : V \longrightarrow (V^*)'$ be given by

$$(w^*)(B_V^\flat v) = B_V(w, v).$$

Then the action of $P \times G(V)$ on $L^2(Hom_D(E, V))$ is given by the formulas

(6) $\qquad \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \cdot \phi(T) = \chi(\mathrm{Tr}_F(XT^* B_V^\flat T))\phi(T), \qquad$ for all $X \in Hom_D(F, E)_{-\varepsilon W}$,

(7) $\qquad \begin{bmatrix} A & \\ & (A^*)^{-1} \end{bmatrix} \cdot \phi(T) = |\det_F(A)|^{-\dim_D(V)/2}\phi(TA), \qquad$ for all $A \in GL(E)$,

(8) $\qquad\qquad\qquad g \cdot \phi(T) = \phi(g^{-1}T), \qquad$ for all $g \in G(V)$.

Let

$$\Omega_V = \{\mathcal{O}_Y \,|\, \mathcal{O}_Y \text{ is open in } Hom_D(E, F)_{-\varepsilon W}, \text{ and } Y = T^* B_V^\flat T \text{ for some } T \in Hom_D(E, V)\}.$$

Given $\mathcal{O}_Y \in \Omega_V$, we will set

$$\Upsilon_Y = \{T \in Hom_D(E, V) \,|\, T^* B_V^\flat T \in \mathcal{O}_Y\}.$$

Then

$$\bigcup_{\mathcal{O}_Y \in \Omega_V} \Upsilon_Y \subset Hom_D(E, V)$$

is a dense open subset, and its complement in $Hom_D(E, V)$ has measure 0. Therefore

(9) $$L^2(Hom_D(E, V)) \cong \bigoplus_{\mathcal{O}_Y \in \Omega_V} L^2(\Upsilon_Y)$$

and each of these spaces is clearly $P \times G(V)$-invariant, according to the formulas given in equations (6)–(8).

We want to show that the spaces $L^2(\Upsilon_Y)$ are equivalent to some induced representation for $P \times G(V)$. To do this, observe that the "geometric" part of the action of $P \times G(V)$ on $L^2(\Upsilon_Y)$ is transitive on Υ_Y. In other words, under the action of $P \times G(V)$ on $Hom_D(E, V)$ given by

$$\left(\begin{bmatrix} A & X \\ & (A^*)^{-1} \end{bmatrix}, g \right) \cdot T = gTA^{-1} \qquad \text{for all } \begin{bmatrix} A & X \\ & (A^*)^{-1} \end{bmatrix} \in P,\ g \in G(V) \text{ and } T \in Hom_D(E, V),$$

each of the Υ_Y's is a single orbit. Fix $T_Y \in \Upsilon_Y$ such that $T_Y^* B_V^\flat T_Y = Y$. The stabilizer of T_Y in $P \times G(V)$ is the subgroup

$$(P \times G(V))_{T_Y} = \left\{ \left(\begin{bmatrix} A & X \\ & (A^*)^{-1} \end{bmatrix}, g \right) \in P \times G(V) \,\middle|\, gT_Y = T_Y A \right\}.$$

Let $g \in G(V)$ be such that $gT_Y = T_Y A$ for some $A \in GL(E)$. Then by the definition of $G(V)$

$$Y = T_Y^* B_V^\flat T_Y = T_Y^* g^* B_V^\flat g T_Y = A^* Y A,$$

that is, A is an element in M_{χ_Y}.

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Define an equivalence relation in $Hom_D(E, V)$ by setting $T \sim S$ if $T = SA$ for some $A \in M_{\chi_Y}$. Given $T \in Hom_D(E, V)$ we will denote its equivalence class, under this equivalence relation, by $[T]$. Let

$$P_{M_{\chi_Y}}(Hom_D(E, V)) = \{[T] \,|\, T \in Hom_D(E, V)\}.$$

Since $G(V)$ acts by left multiplication on $Hom_D(E, V)$, there is natural action of $G(V)$ on the space $P_{M_{\chi_Y}}(Hom_D(E, V))$. Set

$$G(V)_{T_Y} = \{g \in G(V) \,|\, gT_Y = T_Y\} \quad \text{and} \quad G(V)_{[T_Y]} = \{g \in G(V) \,|\, g[T_Y] = [T_Y]\}.$$

Then $(P \times G(V))_{T_Y} \subset M_{\chi_Y} \times G(V)_{[T_Y]}$, and according to equations (6)-(8),

$$(10) \qquad\qquad L^2(\Upsilon_Y) \;\cong\; \mathrm{Ind}_{(P \times G(V))_{T_Y}}^{P \times G(V)} \chi_Y$$

$$(11) \qquad\qquad\qquad \cong\; \mathrm{Ind}_{M_{\chi_Y} N \times G(V)_{[T_Y]}}^{P \times G(V)} \mathrm{Ind}_{(P \times G(V))_{T_Y}}^{M_{\chi_Y} N \times G(V)_{[T_Y]}} \chi_Y$$

Now consider the short exact sequence

$$1 \longrightarrow 1 \times G(V)_{T_Y} \longrightarrow (P \times G(V))_{T_Y} \xrightarrow{q} M_{\chi_Y} N \longrightarrow 1,$$

where q is the projection into the first component. Observe that the map q induces an isomorphism $G(V)_{T_Y} \backslash G(V)_{[T_Y]} \cong M_{\chi_Y}$. From this exact sequence and equation (11), we get that

$$L^2(\Upsilon_Y) \;\cong\; \mathrm{Ind}_{M_{\chi_Y} N \times G(V)_{[T_Y]}}^{P \times G(V)} L^2(G(V)_{T_Y} \backslash G(V)_{[T_Y]})_{\chi_Y}$$

$$(12) \qquad\qquad\qquad \cong\; \mathrm{Ind}_{M_{\chi_Y} N}^{P} L^2(G(V)_{T_Y} \backslash G(V))_{\chi_Y}.$$

The action of $M_{\chi_Y} N$ on $L^2(G(V)_{T_Y} \backslash G(V)_{[T_Y]})_{\chi_Y}$ is given (by definition) as follows: N acts by the character χ_Y, and M_{χ_Y} acts on $L^2(G(V)_{T_Y} \backslash G(V)_{[T_Y]})_{\chi_Y}$ on the left using the isomorphism $G(V)_{T_Y} \backslash G(V)_{[T_Y]} \cong M_{\chi_Y}$. Then according to equations (9) and (12)

$$(13) \qquad\qquad L^2(Hom_D(E, V)) \cong \bigoplus_{\mathcal{O}_Y \in \Omega_V} \mathrm{Ind}_{M_{\chi_Y} N}^{P} L^2(G(V)_{T_Y} \backslash G(V))_{\chi_Y}.$$

But now, from equations (5), (13) and the uniqueness of the decomposition of the N-spectrum, we obtain:

Proposition 3. *As an $M_{\chi_Y} N \times G(V)$-module,*

$$(14) \qquad\qquad L^2(G(V)_{T_Y} \backslash G(V))_{\chi_Y} \cong \int_{\widehat{G}(W)} W_{\chi_Y}(\pi) \otimes \Theta(\pi) \, d\mu_\theta(\pi),$$

Our goal now is to give a more explicit characterization of the spaces $W_{\chi_Y}(\pi)$ and the measure μ_θ appearing in this formula.

3.5. **Stable range.** Let (V, B_V) and (W, B_W) be as before. Assume now that there is a totally isotropic D-submodule $X \subset V$ such that $\dim_D(X) = \dim_D(W)$; in other words, the dual pair $(G(V), G(W))$ is in the *stable range*. In this case, the map

$$\Theta : \widehat{G}(W) \longrightarrow \widehat{G}(V)$$

can be understood in terms of the results of J. S. Li [18]. The measure μ_θ appearing in equation (3) is also known in this case: it is precisely the Plancherel measure of the group $G(W)$. In order to make this paper more self-contained, we will include an alternative calculation of the measure μ_θ using the so-called *mixed model* of the oscillator representation.

3.6. Mixed model. Let X, Y be a totally isotropic, complementary subspaces of V such that $\dim_D(X) = \dim_D(W)$, and let $U = (X \oplus Y)^\perp$. We will use B_V to identify Y with $(X^*)'$ by setting

$$(x^*)y = B_V(x, y), \qquad \text{for all } x \in X,\, y \in Y.$$

Given $A \in GL(X)$, we can use the above identification to define an element $A^* \in GL(Y)$ in the following way: given $x \in X$ and $y \in Y$, we will set $(x^*)(A^*y) := (x^*A^*)y$, i.e., we will define $A^* \in GL(Y)$ by requiring that

$$B_V(x, A^*y) = B_V(Ax, y), \qquad \text{for all } x \in X,\, y \in Y.$$

Observe that the map $A \mapsto (A^*)^{-1}$ defines an isomorphism between $GL(X)$ and $GL(Y)$. Furthermore if $x \in X$, $y \in Y$ and $A \in GL(X)$, then

$$B_V(Ax, (A^*)^{-1}y) = B_V(x, y).$$

Therefore, we can define a map $GL(X) \times G(U) \hookrightarrow G(V)$ that identifies $GL(X) \times G(U)$ with the subgroup of $G(V)$ that preserves the direct sum decomposition $V = X \oplus Y \oplus U$.

Consider the polarization $V \otimes_D W^* = (X \otimes W^* \oplus U \otimes F^*) \bigoplus (Y \otimes W^* \oplus U \otimes E^*)$. Then as a vector space

$$(15) \qquad L^2(X \otimes W^* \oplus U \otimes F^*) \cong L^2(Hom_D(W, X)) \otimes L^2(Hom_D(E, U)).$$

Let $(\omega_U, L^2(Hom_D(E, U)))$ be the Schrödinger model of the oscillator representation associated to the metaplectic group $\widetilde{Sp}(U \otimes_D W^*)$. We will identify the space appearing on the right hand side of equation (15) with the space of L^2 functions from $Hom_D(W, X)$ to $L^2(Hom_D(E, U))$. This is the so called *mixed model* of the oscillator representation.

The action of $G(W) \times GL(X) \times G(U)$ on this model can be described in the following way: If $T \in Hom_D(W, X)$ and $S \in Hom_D(E, U)$, then

$$(16) \qquad g \cdot \phi(T)(S) \;=\; [\omega_U(g)\phi(Tg)](S) \qquad \forall g \in G(W)$$

$$(17) \qquad h \cdot \phi(T)(S) \;=\; \phi(T)(h^{-1}S) \qquad \forall h \in G(U)$$

$$(18) \qquad A \cdot \phi(T)(S) \;=\; |\det_X(A)|^{\dim W/2} \phi(A^{-1}T)(S) \qquad \forall A \in GL(X).$$

We now want to describe this space as an induced representation. To do this, observe that the set of invertible elements in $Hom_D(W, X)$ forms a single orbit under the natural action of $G(W) \times GL(X)$. Furthermore this orbit is open and dense, and its complement has measure 0. Fix $T_0 \in Hom_D(W, X)$ invertible, and define a ε_W-Hermitian form B_{T_0} on X, by setting

$$B_{T_0}(x_1, x_2) = B_W(T_0^{-1}x_1, T_0^{-1}x_2).$$

The group that preserves this form is precisely

$$G(X, B_{T_0}) = \{T_0 g T_0^{-1} \,|\, g \in G(W)\} \subset GL(X).$$

Let

$$(G(W) \times GL(X))_{T_0} = \{(g, T_0 g T_0^{-1}) \,|\, g \in G(W)\} \cong G(W)$$

be the stabilizer of T_0 in $G(W) \times GL(X)$. Then, according to equations (16)–(18),

$$L^2(W^* \otimes X) \otimes L^2(Hom_D(E, U)) \;\cong\; \mathrm{Ind}_{(G(W) \times GL(X))_{T_0}}^{G(W) \times GL(X)} L^2(Hom_D(E, U))$$

$$\cong\; \mathrm{Ind}_{G(W) \times G(X, B_{T_0})}^{G(W) \times GL(X)} \mathrm{Ind}_{(G(W) \times GL(X))_{T_0}}^{G(W) \times G(X, B_{T_0})} L^2(Hom_D(E, U)).$$

Here $(G(W) \times GL(X))_{T_0}$ is acting on $L^2(Hom_D(E, U))$ by taking projection into the first component, and then using the oscillator representation to define an action of $G(W)$ on $L^2(Hom_D(E, U))$. But this representation is equivalent to taking projection into the second component and using the

Schrödinger model of the oscillator representation of $\widetilde{Sp}(U \otimes X^*)$ (where X is equipped with the form B_{T_0}) to define an action of $G(X, B_{T_0})$ on $L^2(Hom_D(T_0(E), U))$. Therefore

$$L^2(W^* \otimes X) \otimes L^2(Hom_D(E, U))$$

$$\cong \quad \operatorname{Ind}_{G(W) \times G(X, B_{T_0})}^{G(W) \times GL(X)} \operatorname{Ind}_{(G(W) \times GL(X))_{T_0}}^{G(W) \times G(X, B_{T_0})} L^2(Hom_D(T_0(E), U))$$

$$\cong \quad \operatorname{Ind}_{G(W) \times G(X, B_{T_0})}^{G(W) \times GL(X)} (\operatorname{Ind}_{(G(W) \times GL(X))_{T_0}}^{G(W) \times G(X, B_{T_0})} 1) \otimes L^2(Hom_D(T_0(E), U))$$

$$\cong \quad \operatorname{Ind}_{G(W) \times G(X, B_{T_0})}^{G(W) \times GL(X)} \int_{\widehat{G}(W)} \pi^* \otimes (\pi^{T_0} \otimes L^2(Hom_D(T_0(E), U))) \, d\mu_{G(W)}(\pi)$$

$$(19) \qquad \cong \quad \int_{\widehat{G}(W)} \pi^* \otimes \operatorname{Ind}_{G(X, B_{T_0})}^{GL(X)} \pi^{T_0} \otimes L^2(Hom_D(T_0(E), U)) \, d\mu_{G(W)}(\pi).$$

Here π^* is the contragredient representation of π, π^{T_0} is the representation of $G(X, B_{T_0})$ given by $\pi^{T_0}(g) = \pi(T_0^{-1} g T_0)$, for all $g \in G(X, B_{T_0})$, and $\mu_{G(W)}$ is the Plancherel measure of $G(W)$. Note that the multiplicity space of π^* in (19) is nonzero for each π in the support of $\mu_{G(W)}$, i.e. as a representation of $G(W)$, Π is weakly equivalent to the regular representation $L^2(G(W))$.

Comparing (3) with (19), we obtain:

Proposition 4. *If $(G(W), G(V))$ is in the stable range, with $G(W)$ the smaller group, then in equations (3) and (14), $\mu_\theta = \mu_{G(W)}$ is the Plancherel measure of $\widehat{G}(W)$.*

3.7. **The Bessel-Plancherel theorem.** Finally, we want to identify the multiplicity space $W_{\chi_Y}(\pi)$ in (14). Note that this is purely an issue about representations of $G(W)$; a priori, it has nothing to do with theta correspondence. What we know is summarized in the following theorem.

Theorem 5 (Bessel-Plancherel theorem). *Let (W, B_W) be an ε_W-Hermitian D-module, and assume that W has a complete polarization $W = E \oplus F$, where E, F are totally isotropic complementary subspaces. Let $P = \{p \in G(W) \,|\, p \cdot E = E\}$ be a Siegel parabolic subgroup of G, and let $P = MN$ be its Langlands decomposition. Given $\chi \in \hat{N}$, let $\mathcal{O}_\chi$ be its orbit under the action of M, and let M_χ be the stabilizer of χ in M. Then*

(1) *For $\mu_{G(W)}$-almost all tempered representations π of $G(W)$,*

$$\pi|_P \cong \bigoplus_{\mathcal{O}_\chi \in \Omega_W} \operatorname{Ind}_{M_\chi N}^P V_\chi(\pi).$$

Here $\mu_{G(W)}$ is the Plancherel measure of $G(W)$, $\Omega_W = \{\mathcal{O}_\chi \in \Omega \,|\, \mathcal{O}_\chi \text{ is open in } \hat{N}\}$, and $V_\chi(\pi)$ is some $M_\chi N$-module such that the action of N is given by the character χ.

(2) *If $\mathcal{O}_\chi \in \Omega_W$, then there is an isomorphism of $M_\chi \times G(W)$-modules:*

$$(20) \qquad L^2(N \backslash G(W); \chi) \cong \int_{\widehat{G}(W)} V_\chi(\pi) \otimes \pi \, d\mu_{G(W)}(\pi).$$

where $V_\chi(\pi)$ is the same space appearing in (1).

(3) *If $\dim_D(W) = 2$, then for $\mathcal{O}_\chi \in \Omega_W$, $\dim V_\chi(\pi) < \infty$ and*

$$V_\chi(\pi) \cong Wh_\chi(\pi) = \{\lambda : \pi^\infty \longrightarrow \mathbb{C} \,|\, \lambda(\pi(n)v) = \chi(n)\lambda(v) \text{ for all } n \in N\}$$

as an $M_\chi N$-module. Here π^∞ stands for the set of smooth vectors of π and the space on the RHS is the space of continuous χ-Whittaker functionals on π^∞.

(4) *If $\mathfrak{k}$ is Archimedean, and M_χ is compact, then*

$$V_\chi(\pi) \subset Wh_\chi(\pi)$$

as a dense subspace, and for any irreducible representation τ of M_χ, one has an equality of τ-isotypic parts:

$$V_\chi(\pi)[\tau] = Wh_\chi(\pi)[\tau].$$

Moreover, this space is finite dimensional.

Proof. Part 2 follows from an argument analogous to the proof of the Whittaker-Plancherel measure given by Sakellaridis-Venkatesh [23, §6.3]. For the proof of part 1 observe that, by the Harish-Chandra Plancherel theorem

$$L^2(G(W))|_{P \times G(W)} = \int_{\widehat{G}(W)} \pi^*|_P \otimes \pi \, d\mu_{G(W)}(\pi).$$

On the other hand

$$L^2(G(W))|_{P \times G(W)} = \bigoplus_{\mathcal{O}_\chi \in \Omega_W} \mathrm{Ind}_{M_\chi N}^P L^2(N \backslash G(W); \chi)$$

$$= \bigoplus_{\mathcal{O}_\chi \in \Omega_W} \mathrm{Ind}_{M_\chi N}^P \int_{\widehat{G}(W)} V_\chi(\pi) \otimes \pi \, d\mu_{G(W)}(\pi)$$

$$= \int_{\widehat{G}(W)} \left[\bigoplus_{\mathcal{O}_\chi \in \Omega_W} \mathrm{Ind}_{M_\chi N}^P V_\chi(\pi) \right] \otimes \pi \, d\mu_{G(W)}(\pi).$$

Therefore

$$\pi^*|_P \cong \bigoplus_{\mathcal{O}_\chi \in \Omega_W} \mathrm{Ind}_{M_\chi N}^P V_\chi(\pi)$$

for $\mu_{G(W)}$-almost all π. In the Archimedean case, this result has also been proved in the thesis of the second named author without the $\mu_{G(W)}$-almost all restriction, yielding an alternative proof of part 2 for the Archimedean case.

Part 3 is part of the Whittaker-Plancherel theorem, which was proved by Wallach in the Archimedean case [27], and independently by Delorme, Sakellaridis-Venkatesh and U-Liang Tang in the p-adic case [4, 23, 25].

Finally, Part 4 was shown by Wallach and the second named author in [11]. □

We note that Theorem 5(1) is a refinement of equation (4): it implies that in (4), only the open orbits $\mathcal{O}_\chi$ in Ω contribute. Moreover, for $\chi \in \Omega_W$, the space $V_\chi(\pi)$ in Theorem 5 is the same as the space $W_\chi(\pi)$ in (4) and (14).

3.8. **Spectral decomposition of generalized Stiefel manifolds.** We may now assemble all the previous results together. For $\mathcal{O}_Y \in \Omega_V$, the space $G(V)_{T_Y} \backslash G(V)$ is known as a *generalized Stiefel manifold*. From equations (14) and (19), we deduce:

Theorem 6. *Suppose that $G(V)_{T_Y} \backslash G(V)$ is a generalized Stiefel manifold. If, in the notation of equation (20)*

$$L^2(N \backslash G(W); \chi_Y) \cong \int_{\widehat{G}(W)} W_{\chi_Y}(\pi) \otimes \pi \, d\mu_{G(W)}(\pi).$$

then

$$L^2(G(V)_{T_Y} \backslash G(V)) \cong \int_{\widehat{G}(W)} W_{\chi_Y}(\pi) \otimes \Theta(\pi) \, d\mu_{G(W)}(\pi).$$

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In a certain sense, the last pair of equations says that the Plancherel measure of the generalized Stiefel manifold $G(V)_{T_Y}\backslash G(V)$ is the pushforward of the Bessel-Plancherel measure of $G(W)$ under the θ-correspondence. We note that in [15], Kobayashi has obtained an explicit description of the discrete spectrum of these generalized Stiefel manifold in the real case, in terms of $A_{\mathfrak{q}}(\lambda)$ modules.

3.9. The Sakellaridis-Venkatesh conjecture.
Using the previous theorem, we can obtain certain examples of the Sakellaridis-Venkatesh conjecture:

- Taking $D = \mathring{k}$, $\mathring{k} \times \mathring{k}$ or $M_2(\mathring{k})$ to be a split $\mathring{k}$-algebra and W to be skew-Hermitian with $\dim_D W = 2$, we obtain the spectral decomposition of $H\backslash G := G(V)_{T_Y}\backslash G(V)$ in terms of the Bessel-Plancherel (essentially the Whittaker-Plancherel) decomposition for G_X, where $H\backslash G$ and G_X are listed in the following table.

$H\backslash G$	$GL_{n-1}\backslash GL_n$	$SO_{n-1}\backslash SO_n$	$Sp_{2n-2}\backslash Sp_{2n}$
G_X	GL_2	$\widetilde{SL_2}$ or SL_2	$SO(4)$

This establishes the cases listed in Table 1 in Theorem 1.

- Taking D to be a quadratic field extension of $\mathring{k}$ or the quaternion division $\mathring{k}$-algebra, and W to be skew-Hermitian, we obtain the spectral decomposition of

$$H\backslash G = U_{n-1}\backslash U_n, \quad Sp_{n-1}(D)\backslash Sp_n(D)$$

in terms of the Bessel-Plancherel decomposition of U_2 and $O_2(D)$. This gives non-split version of the examples above.

3.10. Multiplicity space.
In addition, the multiplicity space $W_\chi(\pi) = Wh_\chi(\pi)$ can be described in terms of the space of H-invariant (continuous) functionals on $\Theta(\pi)^\infty$. Indeed, by the smooth analog of our computation with the Schrodinger model in §3.4, one can show:

Lemma 7. *For any irreducible smooth representation σ^∞ of $G(V)$, let $\Theta^\infty(\sigma^\infty)$ denote the big (smooth) theta lift of σ^∞ to $G(W)$. Then for $\chi \in \mathcal{O}_Y \in \Omega_W \subset \widehat{N}$, there is a natural isomorphism of M_χ-modules:*

$$Wh_\chi(\Theta^\infty(\sigma^\infty)) \cong Hom_{G(V)_{T_Y}}(\sigma^\infty, \mathbb{C}).$$

In the cases we are considering above, one can show that if π is an irreducible tempered representation of $G(W)$, then the small (smooth) theta lift $\sigma = \theta^\infty(\pi^\infty)$ is irreducible (even when $\mathring{k}$ is 2-adic), and moreover, the big (smooth) theta lift $\Theta^\infty(\sigma^\infty)$ of σ^∞ back to $G(W)$ is irreducible and thus isomorphic to π^∞. By our discussion in §3.2, we see that for $\mu_{G(W)}$-almost all π, one has

$$\Theta(\pi)^\infty \cong \theta^\infty(\pi^\infty) = \sigma.$$

Thus the above lemma implies that:

$$W_\chi(\pi) = Wh_\chi(\pi) \cong Hom_H(\Theta(\pi)^\infty, \mathbb{C}).$$

This concludes the proof of the classical cases of Theorem 1.

3.11. Unstable range.
Though we have assumed that $(G(W), G(V))$ is in the stable range from §3.5, it is possible to say something when one is not in the stable range as well. Namely, in §3.6, one would take X to be a maximal isotropic space in V (so $\dim X < \dim W$ here), and consider the mixed model defined on $L^2(Hom_D(W, X)) \otimes L^2(Hom_D(E, U))$. As an illustration, we note the result for the case when W is a symplectic space of dimension 2 and V is a split quadratic space of dimension 3, so that

$$G(W) \times G(V) \cong \tilde{SL}_2 \times SO_3 \cong \tilde{SL}_2 \times PGL_2.$$

For a nonzero $Y \in \widehat{N}$, the subgroup $G(V)_{T_Y}$ of $G(V)$ is simply a maximal torus A_Y of PGL_2.

Proposition 8. *We have*

$$L^2(G(V)_{T_Y}\backslash G(V)) = L^2(A_Y\backslash PGL_2) \cong \int_{\widehat{G(W)}} (W_\chi(\sigma) \otimes W_{\chi_Y}(\sigma)) \otimes \Theta_\chi(\pi) \, d\mu_{G(W)}(\pi).$$

We record the following corollary which is needed in the second half of this paper:

Corollary 9. *The unitary representation $L^2(\mathfrak{sl}_2)$ associated to the adjoint action of PGL_2 on its Lie algebra $\mathfrak{sl}_2$ is weakly equivalent to the regular representation $L^2(PGL_2)$.*

Proof. Since the union of strongly regular semisimple classes are open dense in $\mathfrak{sl}_2$, we see that $L^2(\mathfrak{sl}_2)$ is weakly equivalent to $\bigoplus_A L^2(A\backslash PGL_2)$, where the sum runs over conjugacy classes of maximal tori A in PGL_2. Applying Proposition 8, one deduces that

$$\bigoplus_A L^2(A\backslash PGL_2) \cong \int_{\widehat{G(W)}} M_\chi(\pi) \otimes \Theta_\chi(\pi) \, d\mu_{G(W)}(\pi)$$

with

$$M_\chi(\pi) = W_\chi(\pi) \otimes \left(\bigoplus_A W_{\chi_A}(\pi) \right).$$

One can show that the theta correspondence with respect to χ induces a bijection

$$\Theta_\chi : \{\pi \in \widehat{G(W)} : W_\chi(\pi) \neq 0\} \longleftrightarrow \widehat{G(V)}.$$

Moreover, one can write down this bijection explicitly (in terms of the usual coordinates on the unitary duals of $\widetilde{SL}_2$ and PGL_2). From this description, one sees that

$$(\Theta_\chi)_*(\mu_{G(W)}) = \mu_{G(V)}.$$

This shows that

$$\int_{\widehat{G(W)}} M_\chi(\pi) \otimes \Theta_\chi(\pi) \, d\mu_{G(W)}(\pi) \cong \int_{\widehat{G(V)}} M_\chi(\Theta_\chi^{-1}(\sigma)) \otimes \sigma \, d\mu_{G(V)}(\sigma),$$

with $M_\chi(\Theta_\chi^{-1}(\sigma)) \neq 0$. This proves the corollary. $\square$

4. Exceptional Structures and Groups

The argument of the previous section can be adapted to various dual pairs in exceptional groups, thus giving rise to more exotic examples of the Sakellaridis-Venkatesh conjecture. In particular, we shall show that the spectral decomposition of $L^2(X) = L^2(H\backslash G)$ can obtained from that of $L^2(G_X)$, with X and G_X given in the following table.

X	$SO_3\backslash SL_3$	$SL_3\backslash G_2$	$(J,\psi)\backslash G_2$	$Sp_6\backslash SL_6$	
G_X	$\widetilde{SL}_3$	SL_2	PGL_3	SL_3	
X	$SU_3\backslash Spin_7$	$G_2\backslash Spin_7$	$G_2\backslash Spin_8$	$Spin_9\backslash F_4$	$F_4\backslash E_6$
G_X	$(Spin_3 \times Spin_5)/\Delta\mu_2$	SL_2	$SL_2^3/\Delta\mu_2$	PGL_2	SL_3

$$\text{TABLE } 3$$

The unexplained notation will be explained in due course. Comparing with the tables in [23, §15 and §16], we see that these exceptional examples, together with the classical examples treated earlier, verify the conjecture of Sakallaridis-Venkatesh for almost all the rank 1 spherical varieties (with

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certain desirable properties), and also some rank 2 or rank 3 ones. Indeed, they also include low rank examples of several infinite families of spherical varieties, such as $Sp_{2n} \backslash SL_{2n}$, $SO_n \backslash SL_n$ and $SU_n \backslash SO_{2n+1}$.

Though the proof will be similar in spirit to that of the previous section, we shall need to deal with the geometry of various exceptional groups, and this is ultimately based on the geometry of the (split) octonion algebra $\mathbb{O}$ and the exceptional Jordan algebra $J(\mathbb{O})$. Thus we need to recall some basic properties of $\mathbb{O}$ and its automorphism group. A good reference for the material in this section is the book [13]. One may also consult [19] and [28].

4.1. **Octonions and G_2.** Let $\hat{k}$ be a local field of characteristic zero and let $\mathbb{O}$ denote the (8-dimensional) split octonion algebra over $\hat{k}$. The octonion algebra $\mathbb{O}$ is non-commutative and non-associative. Like the quaternion algebra, it is endowed with a conjugation $x \mapsto \bar{x}$ with an associated trace map $Tr(x) = x + \bar{x}$ and an associated norm map $N(x) = x \cdot \bar{x}$. It is a composition algebra, in the sense that $N(x \cdot y) = N(x) \cdot N(y)$.

A useful model for $\mathbb{O}$ is the so-called Zorn's model, which consists of 2×2-matrices

$$\begin{pmatrix} a & v \\ v' & b \end{pmatrix}, \quad \text{with } a, b \in \hat{k}, \ v \in V \cong k^3 \text{ and } v' \in V',$$

with V a 3-dimensional $\hat{k}$-vector space with dual V'. By fixing an isomorphism $\wedge^3 V \cong \hat{k}$, one deduces natural isomorphisms

$$\wedge^2 V \cong V' \quad \text{and} \quad \wedge^2 V' \cong V,$$

and let $\langle -, - \rangle$ denote the natural pairing on $V' \times V$. The multiplication on $\mathbb{O}$ is then defined by

$$\begin{pmatrix} a & v \\ v' & b \end{pmatrix} \cdot \begin{pmatrix} c & w \\ w' & d \end{pmatrix} = \begin{pmatrix} ac + \langle w', v \rangle & aw + dv + v' \wedge w' \\ cv' + bw' + v \wedge w & bd + \langle v', w \rangle \end{pmatrix}$$

The conjugation map is

$$\begin{pmatrix} a & v \\ v' & b \end{pmatrix} \mapsto \begin{pmatrix} b & -v \\ -v' & a \end{pmatrix}$$

so that

$$Tr \begin{pmatrix} a & v \\ v' & b \end{pmatrix} = a + b \quad \text{and} \quad N \begin{pmatrix} a & v \\ v' & b \end{pmatrix} = ab - \langle v', v \rangle.$$

Any non-central element $x \in \mathbb{O}$ satisfies the quadratic polynomial $x^2 - Tr(x) \cdot x + N(x) = 0$. Thus, a non-central element $x \in \mathbb{O}$ generates a quadratic $\hat{k}$-subalgebra described by this quadratic polynomial. If this quadratic polynomial is separable, x is said to have rank 2. Otherwise, x is said to have rank 1.

The automorphism group of the algebra $\mathbb{O}$ is the split exceptional group of type G_2. The group G_2 contains the subgroup $SL(V) \cong SL_3$ which fixes the diagonal elements in Zorn's model, and acts on V and V' naturally. Clearly, G_2 fixes the identity element $1 \in \mathbb{O}$, so that it acts on the subspace $\mathbb{O}_0$ of trace zero elements. The following proposition summarizes various properties of the action of G_2 on $\mathbb{O}_0$.

Proposition 10. *(i) Fix $a \in \hat{k}^\times$, and let Ω_a denote the subset of $x \in \mathbb{O}_0$ with $N(x) = a$, then Ω_a is nonempty and G_2 acts transitively on Ω_a with stabilizer isomorphic to $SU_3(E_a)$, where $E_a = \hat{k}[x]/(x^2 - a)$.*

(ii) The automorphism group G_2 acts transitively on the set Ω_0 of trace zero, rank 1 elements. For $x \in \Omega_0$, the stabilizer of the line $\hat{k} \cdot x$ is a maximal parabolic subgroup $Q = L \cdot U$ with Levi factor $L \cong GL_2$ and unipotent radical U a 3-step unipotent group.

Now we note:

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- When $a \in (\hat{k}^{\times})^2$ in (i), the stabilizer of an element in Ω_a is isomorphic to SL_3; this explains the 2nd entry in Table 3.
- In (ii), the 3-step filtration of U is given by

$$U \supset [U,U] \supset Z(U) \supset \{1\}$$

  where $[U,U]$ is the commutator subgroup and $Z(U)$ is the center of U. Moreover,

$$\dim Z(U) = 2 \quad \text{and} \quad \dim[U,U] = 3,$$

  so that $[U,U]/Z(U) \cong \hat{k}$. If ψ is a non-trivial character of $\hat{k}$, then ψ gives rise to a nontrivial character of $[U,U]$ which is fixed by the subgroup $[L,L] \cong SL_2$. Setting $J = [L,L] \cdot [U,U]$, we may extend ψ to a character of J trivially across $[L,L]$. This explains the 3rd entry of Table 3.

Though the octonionic multiplication is neither commutative or associative, the trace form satisfies:

$$Tr((x \cdot y) \cdot z) = Tr(x \cdot (y \cdot z)),$$

(so there is no ambiguity in denoting this element of $\hat{k}$ by $Tr(x \cdot y \cdot z)$) and G_2 is precisely the subgroup of $SO(\mathbb{O}, N)$ satisfying

$$Tr((gx) \cdot (gy) \cdot (gy)) = Tr(x \cdot y \cdot z) \quad \text{for all } x, y, z \in \mathbb{O}.$$

4.2. **Exceptional Jordan algebra and F_4.** Let $J = J(\mathbb{O})$ denote the 27-dimensional vector space consisting of all 3×3 Hermitian matrices with entries in $\mathbb{O}$. Then a typical element in J has the form

$$\alpha = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix}, \quad \text{with } a, b, c \in \hat{k} \text{ and } x, y, z \in \mathbb{O}.$$

The set J is endowed with a multiplication

$$\alpha \circ \beta = \frac{1}{2} \cdot (\alpha\beta + \beta\alpha)$$

where the multiplication on the RHS refers to usual matrix multiplication. With this multiplication, J is the exceptional Jordan algebra.

The algebra J carries a natural cubic form $d = \det$ given by the determinant map on J, and a natural linear form tr given by the trace map. Moreover, every element in J satisfies a cubic polynomial, by the analog of the Cayley-Hamilton theorem. An element $\alpha \in J$ is said to be of rank n if its minimal polynomial has degree n, so that $0 \le n \le 3$. For example, $\alpha \in J$ has rank 1 if and only if its entries satisfy

$$N(x) = bc, \ N(y) = ca, \ N(z) = ab, \ xy = c\bar{z}, \ yz = a\bar{x}, \ zx = b\bar{y}.$$

More generally, the above discussion holds if one uses any composition $\hat{k}$-algebra in place of $\mathbb{O}$. Thus, if $B = \hat{k}$, a quadratic algebra K, a quaternion algebra D or the octonion algebra $\mathbb{O}$, one has the Jordan algebra $J(B)$. One may consider the group $\mathrm{Aut}(J(B), \det)$ of invertible linear maps on $J(B)$ which fixes the cubic form $\det$, and its subgroup $\mathrm{Aut}(J, \det, e)$ which fixes an element e with $\det(e) \ne 0$. For the various B's, these groups are listed in the following table.

Proposition 11. (i) For any $a \in \hat{k}^{\times}$, the group $\mathrm{Aut}(J(B), \det)$ acts transitively on the set of $e \in J$ with $\det(e) = a$, with stabilizer group $\mathrm{Aut}(J(B), \det, e)$ described in the above table. If e is the unit element of $J(B)$, then $\mathrm{Aut}(J(B), \det, e)$ is the automorphism group of the Jordan algebra $J(B)$.

B	$\mathring{\mathfrak{k}}$	K	D	$\mathbb{O}$
$\mathrm{Aut}(J(B),\det)$	SL_3	$SL_3(K)/\Delta\mu_3$	$SL_3(D)/\mu_2 = SL_6/\mu_2$	E_6
$\mathrm{Aut}(J(B),\det,e)$	SO_3	SL_3	$PGSp_6$	F_4

TABLE 4

(ii) The group $F_4 = \mathrm{Aut}(J(\mathbb{O}))$ acts transitively on the set of rank 1 elements in $J(\mathbb{O})$ of trace $a \neq 0$. The stabilizer of a point is isomorphic to the group $Spin_9$ of type B_4.

In particular, the proposition explains the 1st, 4th, 8th and 9th entry of Table 3.

4.3. **Triality and $Spin_8$.** An element $\alpha \in J = J(\mathbb{O})$ of rank 3 generates a commutative separable cubic subalgebra $\mathring{\mathfrak{k}}(\alpha) \subset J$. For any such cubic F-algebra E, one may consider the set Ω_E of algebra embeddings $E \hookrightarrow J$. Then one has:

Proposition 12. *(i) The set Ω_E is non-empty and the group F_4 acts transitively on Ω_E.*

(ii) The stabilizer of a point in Ω_E is isomorphic to the quasi-split simply-connected group $Spin_8^E$ of absolute type D_4.

(iii) Fix an embedding $j : E \hookrightarrow J$ and let $E^\perp$ denote the orthogonal complement of the image of E with respect to the symmetric bilinear form $(\alpha, \beta) = tr(\alpha \circ \beta)$. The action of the stabilizer $Spin_8^E$ of j on $E^\perp$ is the 24-dimensional Spin representation, which on extending scalars to $\bar{\mathfrak{k}}$, is the direct sum of the three 8-dimensional irreducible representations of $Spin_8(\bar{\mathfrak{k}})$ whose highest weights correspond to the 3 satellite vertices in the Dynkin diagram of type D_4.

As an example, suppose that $E = \mathring{\mathfrak{k}} \times \mathring{\mathfrak{k}} \times \mathring{\mathfrak{k}}$, and we fix the natural embedding $E \hookrightarrow J$ whose image is the subspace of diagonal elements in J. Then $E^\perp$ is naturally $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, and the split group $Spin_8$ acts on this, preserving each copy of $\mathbb{O}$. This gives an injective homomorphism

$$\rho : Spin_8 \longrightarrow SO(\mathbb{O}, N) \times SO(\mathbb{O}, N) \times SO(\mathbb{O}, N)$$

whose image is given by

$$Spin_8 \cong \{g = (g_1, g_2, g_3) : Tr((g_1 x) \cdot (g_2 y) \cdot (g_3 z)) = Tr(x \cdot y \cdot z) \quad \text{for all } x, y, z \in \mathbb{O}\}.$$

From this description, one sees that there is an action of $\mathbb{Z}/3\mathbb{Z}$ on $Spin_8$ given by the cyclic permutation of the components of g, and the subgroup fixed by this action is precisely

$$G_2 = Spin_8^{\mathbb{Z}/3\mathbb{Z}}.$$

This explains the 7th entry of Table 3.

More generally , the stabilizer of a triple $(x, y, z) \in \mathbb{O}^3$ with $(x \cdot y) \cdot z \in \mathring{\mathfrak{k}}^\times$ is a subgroup of $Spin_8$ isomorphic to G_2 (see [28]). For example, the stabilizer in $Spin_8$ of the vector $(1, 0, 0) \in \mathbb{O}^3$ is isomorphic to the group $Spin_7$ which acts naturally on $\mathbb{O}_0 \oplus \mathbb{O} \oplus \mathbb{O}$. The action of $Spin_7$ on $\mathbb{O}_0$ is via the standard representation of SO_7, whereas its action on the other two copies of $\mathbb{O}$ is via the Spin representation. From the discussion above, we see that the stabilizer in $Spin_7$ of $(x, \bar{x}) \in \mathbb{O}^2$, with $N(x) \neq 0$, is isomorphic to the group G_2. In particular, this explains the 6th entry of Table 3.

On the other hand, the stabilizer in $Spin_8$ of a triple $(x, y, z) \in \mathbb{O}^3$ with $(x \cdot y) \cdot z \notin \mathring{\mathfrak{k}}^\times$ is isomorphic to $SU_3 \subset G_2 \subset Spin_7 \subset Spin_8$ (see [28]). For example, if one takes $x = y = 1 \in \mathbb{O}$ and $z \notin \mathring{\mathfrak{k}}$, then $K = \mathring{\mathfrak{k}}[z]$ is an étale quadratic subalgebra of $\mathbb{O}$ and it follows by Proposition 10 that the stabilizer of $(1, 1, z)$ is isomorphic to $SU_3(K) \subset Spin_7$. This explains the 5th entry in Table 3.

By the above discussion, it is not difficult to show:

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Proposition 13. *(i) The group $Spin_8$ acts transitively on the set of rank 1 elements*

$$\alpha = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \in J(\mathbb{O})$$

with diagonal part $(a, b, c) \in \mathfrak{k}^\times \times \mathfrak{k}^\times \times \mathfrak{k}^\times$ fixed. Moreover, the stabilizer of a point is isomorphic to G_2.

(ii) The group $Spin_7$ acts transitively on the set of rank 1 elements $\alpha \in J(\mathbb{O})$ as in (i) above, with a, b, c and $Tr(x) \in \mathfrak{k}$ fixed and $a \cdot (bc - Tr(x)) \neq 0$.

4.4. $SL_3 \backslash G_2$ **and** $G_2 \backslash Spin_7$. From the discussion above, we see that there are isomorphisms of homogeneous varieties

$$SL_3 \backslash G_2 \cong SO_6 \backslash SO_7 \quad \text{and} \quad G_2 \backslash Spin_7 \cong Spin_7 \backslash Spin_8 \cong SO_7 \backslash SO_8.$$

Since we have already determined the spectral decomposition of $L^2(SO_6 \backslash SO_7)$ and $L^2(SO_7 \backslash SO_8)$ in terms of the spectral decomposition of $L^2(\tilde{SL}_2)$ and $L^2(SL_2)$ respectively, we obtain the desired description for $SL_3 \backslash G_2$ and $G_2 \backslash Spin_7$. We note that in [16], Kobayashi used the same observation to deduce the Plancherel theorem of these spherical varieties from the Plancherel theorem for the corresponding symmetric spaces given above in the real case. He also gave an explicit description of the branching from SO_7 to G_2 (for representations of SO_7 occurring in $L^2(SO_6 \backslash SO_7)$) in the real case; the p-adic case of this branching is shown in [8].

The rest of the paper is devoted to the remaining cases in Table 3.

5. **Exceptional Dual Pairs**

In this section, we introduce some exceptional dual pairs contained in the adjoint groups of type F_4, E_6, E_7 and E_8. We begin with a uniform construction of the exceptional Lie algebras of the various exceptional groups introduced above. This construction can be found in [22] and will be useful for exhibiting various reductive dual pairs. The reader may consult [19], [22], [24] and [28] for the material of this section.

5.1. **Exceptional Lie algebras.** Consider the chain of Jordan algebras

$$\mathfrak{k} \subset \mathfrak{k} \times \mathfrak{k} \subset E \subset J(\mathfrak{k}) \subset J(K) \subset J(D) \subset J(\mathbb{O})$$

where E is a cubic $\mathfrak{k}$-algebra, K a quadratic $\mathfrak{k}$-algebra and D a quaternion $\mathfrak{k}$-algebra, and one has the containment $\mathfrak{k} \times \mathfrak{k} \subset E$ only when $E = \mathfrak{k} \times K$ is not a field. Denoting such an algebra by $\mathcal{R}$, the determinant map det of $J(\mathbb{O})$ restricts to give a cubic form on $\mathcal{R}$. Now set

(21) $$\mathfrak{s}_\mathcal{R} = \mathfrak{sl}_3 \oplus \mathfrak{m}_\mathcal{R} \oplus (\mathfrak{k}^3 \otimes \mathcal{R}) \oplus (\mathfrak{k}^3 \otimes \mathcal{R})',$$

with

$$\mathfrak{m}_\mathcal{R} = \mathrm{Lie}(Aut(\mathcal{R}, \det)).$$

One can define a Lie algebra structure on $\mathfrak{s}_\mathcal{R}$ [22] whose type is given by the following table.

$\mathcal{R}$	$\mathfrak{k}$	$\mathfrak{k} \times \mathfrak{k}$	E	$J(\mathfrak{k})$	$J(K)$	$J(D)$	$J(\mathbb{O})$
$\mathfrak{m}_\mathcal{R}$	0	$\mathfrak{k}$	E_0	$\mathfrak{sl}_3$	$\mathfrak{sl}_3(K)$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathfrak{s}_\mathcal{R}$	$\mathfrak{g}_2$	$\mathfrak{b}_3$	$\mathfrak{d}_4$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

We denote the corresponding adjoint group with Lie algebra $\mathfrak{s}_\mathcal{R}$ by $S_\mathcal{R}$, or simply by S if $\mathcal{R}$ is fixed and understood.

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathfrak{k}^3$ with dual basis $\{e_i'\}$. The subalgebra of $\mathfrak{sl}_3$ stabilizing the lines $\mathfrak{k}e_i$ is the diagonal torus $\mathfrak{t}$. The nonzero weights under the adjoint action of $\mathfrak{t}$ on $\mathfrak{s}_\mathcal{R}$ form

a root system of type G_2. The long root spaces are of dimension 1 and are precisely the root spaces of $\mathfrak{sl}_3$, i.e. the spaces spanned by $e_i' \otimes e_j$. We shall label these long roots by β, β_0 and $\beta_0 - \beta$, with corresponding 1-parameter subgroups

$$u_\beta(x) = \begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad u_{\beta_0}(x) = \begin{pmatrix} 1 & 0 & x \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad u_{\beta_0 - \beta}(x) = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & x \\ & & 1 \end{pmatrix}$$

We also let

$$w_\beta = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

denote the Weyl group element associated to β. The short root spaces, on the other hand, are $e_i \otimes \mathcal{R}$ and $e_i' \otimes \mathcal{R}'$ and are thus identifiable with $\mathcal{R}$.

5.2. **Exceptional dual pairs.** We can now exhibit 2 families of dual pairs in $S_\mathcal{R}$.

- From (21), one has

$$\mathfrak{sl}_3 \oplus \mathfrak{m}_\mathcal{R} \subset \mathfrak{s}_\mathcal{R}.$$

This gives a family of dual pairs

(22) $$SL_3 \times Aut(\mathcal{R}, \det) \longrightarrow S_\mathcal{R}.$$

We shall only be interested in these dual pairs when $\mathcal{R} = J(B)$.

- For a pair of Jordan algebras $\mathcal{R}_0 \subset \mathcal{R}$, we have $\mathfrak{s}_{\mathcal{R}_0} \subset \mathfrak{s}_\mathcal{R}$ which gives a subgroup $G_{\mathcal{R}_0} \subset S_\mathcal{R}$, where $G_{\mathcal{R}_0}$ is isogeneous to $S_{\mathcal{R}_0}$. If $G'_{\mathcal{R}_0,\mathcal{R}} = Aut(\mathcal{R}, \mathcal{R}_0)$, then one has a second family of dual pairs

(23) $$G_{\mathcal{R}_0} \times G'_{\mathcal{R}_0,\mathcal{R}} \longrightarrow S_\mathcal{R}.$$

With $\mathcal{R}_0 \subset \mathcal{R}$ fixed, we shall simply write $G \times G'$ for this dual pair. For the various pairs $\mathcal{R}_0 \subset \mathcal{R}$ of interest here, we tabulate the associated dual pairs in the table below.

$\mathcal{R}_0 \subset \mathcal{R}$	$\Bbbk \subset J(K)$	$\Bbbk \times \Bbbk \subset J(D)$	$E \subset J(D)$	$J(\Bbbk) \subset J(D)$
$G \times G'$	$G_2 \times PGL_3$	$Spin_7 \times (Spin_3 \times Spin_5)/\Delta\mu_2$	$Spin_8 \times SL_2(E)/\Delta\mu_2$	$F_4 \times PGL_2$

Observe that in the language of Table 3, with $X = H \backslash G$, the dual pairs described above are precisely $G_X \times G$.

5.3. **Heisenberg parabolic.** The presentation (21) also allows one to describe certain parabolic subalgebras of $\mathfrak{s}_\mathcal{R}$. If we consider the adjoint action of

$$t = \operatorname{diag}(1, 0, -1) \in \mathfrak{sl}_3$$

on $\mathfrak{s}$, we obtain a grading $\mathfrak{s} = \oplus_i \mathfrak{s}[i]$ by the eigenvalues of t. Then

$$\begin{cases} \mathfrak{s}[0] = \mathfrak{t} \oplus \mathfrak{m} \oplus (e_2 \otimes \mathfrak{R}) \oplus (e_2' \otimes \mathcal{R}') \\ \mathfrak{s}[1] = \Bbbk e_2' \otimes e_1 \oplus (e_1 \otimes \mathcal{R}) \oplus (e_3' \otimes \mathcal{R}') \oplus \Bbbk e_3' \otimes e_2 \\ \mathfrak{s}[2] = \Bbbk e_3' \otimes e_1, \end{cases}$$

and $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{s}[i]$ is a Heisenberg parabolic subalgebra.

We denote the corresponding Heisenberg parabolic subgroup by $P_S = M_S \cdot N_S$. In particular, its unipotent radical is a Heisenberg group with 1-dimensional center $Z_S \cong u_{\beta_0}(\Bbbk) \cong \mathfrak{s}[2]$ and

$$N_S/Z_S \cong \mathfrak{s}[1] = \Bbbk \oplus \mathcal{R} \oplus \mathcal{R}' \oplus \Bbbk,$$

The semisimple type of its Levi factor M_S is given in the table below.

S	F_4	E_6	E_7	E_8
M_S	C_3	A_5	D_6	E_7

The Lie bracket defines an alternating form on N_S/Z_S which is fixed by $P_S^1 = [P_S, P_S]$. This gives an embedding

$$P_S^1 = M_S^1 \cdot N_s \hookrightarrow Sp(N_S/Z_S) \ltimes N_S.$$

5.4. **Intersection with dual pairs.** For a pair $\mathcal{R}_0 \subset \mathcal{R}$, with associated dual pair given in (23), it follows by construction that

$$(G_{\mathcal{R}_0} \times G'_{\mathcal{R}_0,\mathcal{R}}) \cap P_S = P \times G'_{\mathcal{R}_0,\mathcal{R}},$$

where P is the Heisenberg parabolic subgroup of $G_{\mathcal{R}_0}$. On the other hand, for the family of dual pairs given in (22),

$$(SL_3 \times Aut(\mathcal{R}, \det)) \cap P_S = B \times Aut(\mathcal{R}, \det)$$

where B is a Borel subgroup of SL_3.

5.5. **Siegel parabolic.** The group S of type E_6 or E_7 has a Siegel parabolic subgroup $Q_S = L_S \cdot U_S$ whose unipotent radical U_S is abelian; we call this a Siegel parabolic subgroup. The semisimple type of L_S and the structure of U_S as an L_S-module is summarized in the following table.

S	L_S	U_S	U_S as L_S-module
E_6	D_5	$\mathbb{O} \oplus \mathbb{O}$	half spin representation of dimension 16
E_7	E_6	$J(\mathbb{O})$	miniscule representation of dimension 27

Let $\Omega_Q \subset \overline{U}_S$ be the orbit of a highest weight vector in $\overline{U}_S$. The following proposition describes the set Ω_Q:

Proposition 14. *(i) If S is of type E_6, then*

$$\Omega_Q = \{(x, y) \in \mathbb{O}^2 : N(x) = N(y) = 0 = x \cdot \bar{y}\}.$$

(ii) If S is of type E_7, then

$$\Omega_Q = \{\alpha \in J : rank(\alpha) = 1\}.$$

5.6. **Intersection with dual pairs.** With $\mathcal{R}_0 \subset \mathcal{R}$ fixed, with associated dual pair $G \times G'$ as given in (23), one may choose Q_S so that

$$(G \times G') \cap Q_S = G \times Q_0$$

with $Q_0 = L_0 \cdot U_0$ a Siegel parabolic subgroup of G', so that U_0 is abelian. The group Q_0 and the embedding $U_0 \subset U_S$ can be described by the following table.

G'	PGL_3	$(Spin_3 \times Spin_5)/\Delta\mu_2$	$SL_2(E)/\Delta\mu_2$	PGL_2
Q_0	maximal parabolic	(Borel)$\times$ (Siegel parabolic)	Borel	Borel
$U_0 \subset U_S$	$\hat{k}^2 \subset \mathbb{O}^2$	$\hat{k} \oplus Sym^2(\hat{k}) \subset J(\mathbb{O})$	$E \subset J(\mathbb{O})$	$\hat{k} \subset J(\mathbb{O})$

Identifying the opposite unipotent radical $\bar{U}_0$ with the dual space of U_0 using the Killing form, one has a natural projection

$$\tau : \bar{U}_S \longrightarrow U_0.$$

This is simply given by the projection from U_S to U_0 along $U_0^{\perp}$.

6. Generic Orbits

In this section, we consider an orbit problem which will be important for our applications. Namely, with the notation at the end of the last section, we have an action of $L_0 \times G$ on the set $\Omega_Q \subset \bar{U}_S$. We would like to determine the generic orbits of this action. For simplicity, we shall consider the case when $S = E_6$ and E_7 separately.

6.1. Dual Pair in E_6.

Suppose first that $S = E_6$ so that $G' \times G = PGL_3 \times G_2$. In this case, the natural $L \times G_2$-equivariant projection $\tau : \bar{U}_S \longrightarrow \bar{U}_0$ is given by

$$\tau(x, y) = (Tr(x), Tr(y)).$$

The nonzero elements in $\bar{U}_0 \cong \hat{k}^2$ are in one orbit of L_0; we fix a representative $(0, 1) \in \hat{k}^2$ and note that its stabilizer in L_0 is the "mirabolic" subgroup P_{L_0} of $L_0 \cong GL_2$. Then the fiber over $(0, 1)$ is given by

$$\{(x, y) \in \mathbb{O}^2 : N(x) = N(y) = Tr(x) = 0, \, Tr(y) = 1, \, x \cdot \bar{y} = 0\},$$

and carries a natural action of $P_{L_0} \times G_2$. We note:

Lemma 15. *(i) The group G_2 acts transitively on the fiber $\tau^{-1}(0, 1)$ and the stabilizer of a point (x_0, y_0) is isomorphic to the subgroup $[L, L] \cdot Z(U) \subset J$.*

(ii) If we consider the subset $\{(x_0, y_0 + \lambda x_0) : \lambda \in \hat{k}\} \subset \tau^{-1}(0, 1)$, then the subgroup of $P_{L_0} \times G_2$ stabilizing this subset is isomorphic to

$$(P_{L_0} \times L \cdot [U, U])^0 = \{(h, g \cdot u) : \det h = \det g\}.$$

The action of the element

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \times g \cdot u \in (P_{L_0} \times L \cdot [U, U])^0$$

is by

$$(x_0, y_0 + \lambda x_0) \mapsto (x_0, y_0 + a^{-1} \cdot (\lambda + b - p(u))x_0)$$

where $p : J \longrightarrow \hat{k} \cong J/[L, L] \cdot Z(U)$ is the natural projection. Thus, there is a unique generic $L_0 \times G_2$ orbit on Ω_Q given by

$$(L_0 \times G_2) \times_{(P_{L_0} \times L \cdot [U,U])^0} \hat{k}.$$

6.2. Dual Pairs in E_7.

Now suppose that $S = E_7$. As above, we first determine the generic L_0-orbits on $\bar{U}_0$. For each generic L_0-orbit in $\bar{U}_0$, let us take a representative χ and let Z_χ denote its stabilizer in L_0. Then the fiber $\tau^{-1}(\chi)$ is preserved by $Z_\chi \times G$. In each case, it follows by Prop. 11(ii) and Prop. 13 that G acts transitively on $\tau^{-1}(\chi)$. Denote the stabilizer in G of $\tilde{\chi} \in \tau^{-1}(\chi)$ by H_χ. Then under the action of $Z_\chi \times G$, the stabilizer group $\tilde{H}_\chi$ of $\tilde{\chi}$ sits in a short exact sequence

$$1 \longrightarrow H_\chi \longrightarrow \tilde{H}_\chi \xrightarrow{\ p\ } Z_\chi \longrightarrow 1.$$

Thus, the generic $L_0 \times G$-orbits are given by the disjoint union

$$\bigcup_{\text{generic } \chi} (Z_\chi \times G) \times_{\tilde{H}_\chi} \tilde{\chi}$$

where the union runs over the generic L_0-orbits on $\bar{U}_0$ and $\tilde{\chi}$ is an element in $\tau^{-1}(\chi)$ with stabilizer $\tilde{H}_\chi$. We summarize this discussion in the following table.

$G \times G'$	$F_4 \times PGL_2$	$Spin_8 \times SL_2(E)/\Delta\mu_2$	$Spin_7 \times (Spin_3 \times Spin_5)/\Delta\mu_2$
generic L_0-orbits	singleton	(a,b,c) $\in (\hat{\mathfrak{k}}^\times/\hat{\mathfrak{k}}^{\times 2})^3/\Delta\hat{\mathfrak{k}}^\times$	$A = diag(b,c)$ $\in Sym^2\hat{\mathfrak{k}}^2/GL_2(\hat{\mathfrak{k}})$
$\tau^{-1}(\chi)$	$\alpha \in J(\mathbb{O})$ rank 1, trace 1	$\alpha \in J(\mathbb{O})$ diagonal= (a,b,c)	$\alpha \in J(\mathbb{O})$ diagonal= $(1,b,c)$, $Tr(z)=0$
Z_χ	trivial	center of $G' = \mu_2 \times \mu_2$	O_2
H_χ	$Spin_9$	G_2	SU_3

7. Minimal Representation

In this section, we introduce the (unitary) minimal representation Π of S and describe some models for Π. Note that when $S = F_4$, Π is actually a representation of the double cover of F_4. When S is of type E, then Π is a representation of S.

7.1. Schrodinger model.

Because the groups $S = E_6$ and E_7 have a Siegel parabolic subgroup, there is an analog of the Schrodinger model for the minimal representation Π of S. By [6], the representation Π can be realized on the space $L^2(\Omega_Q, \mu_Q)$ of square-integrable functions on Q with respect to a L_S-equivariant measure μ_Q on Ω_Q. This is analogous to the Schrodinger model of the Weil representation. In particular, we have the following action of Q_S on Π:

$$\begin{cases} (l \cdot f)(\chi) = \delta_{Q_S}(l)^r \cdot f(l^{-1} \cdot \chi) \\ (u \cdot f)(\chi) = \chi(u) \cdot f(\chi), \end{cases}$$

where $r = 1/4$ (resp. $2/9$) if S is of type E_6 (resp. E_7).

7.2. Mixed model.

For general $S = S_{\mathcal{R}}$, one has the analog of the mixed model, on which the action of the Heisenberg group P_S is quite transparent. Recall that $N_S/Z_S = \hat{\mathfrak{k}} \oplus \mathcal{R} \oplus \mathcal{R}' \oplus \hat{\mathfrak{k}}$ and one has an embedding

$$P_S^1 = [P_S, P_S] \hookrightarrow \mathrm{Sp}(N_S/Z_S) \ltimes N_S.$$

Then by [14], the mixed model of the minimal representation is realized on the Hilbert space

$$\mathrm{Ind}_{P_S^1}^{P_S} L^2(\mathcal{R}' \oplus \hat{\mathfrak{k}}') \cong L^2(\hat{\mathfrak{k}}^\times \oplus \mathcal{R} \oplus \hat{\mathfrak{k}}),$$

where the action of P_S^1 on $L^2(\mathcal{R} \oplus \hat{\mathfrak{k}})$ is via the Heisenberg-Weil representation (associated to any fixed additive character ψ of $\hat{\mathfrak{k}}$). The explicit formula can be found in [22, Prop. 43].

In fact, one can describe the full action of S on Π by giving the action of an extra Weyl group element. More precisely, if w_β is the standard Weyl group element in SL_3 associated to the root β (see §5.1), then by [22, Prop. 47], one has

$$(w_\beta \cdot f)(t, x, a) = \psi(\det(x)/a) \cdot f(-a/t, x, -a).$$

Since S is generated by P_S and the element w_0, this completely determines the representation Π.

For example, one may work out the action of an element $u_{-\beta}(b) = w_\beta u_\beta(b) w_\beta^{-1}$ (see §5.1). A short computation gives:

$$(u_{-\beta}(b) \cdot f)(t, x, a) = \psi\left(\frac{b \det(x)}{a - t^2}\right) \cdot f(t - \frac{ab}{t}, a - \frac{a^2 b}{t^2}, x).$$

If f is continuous, then the above formula gives:

$$(24) \qquad (u_{-\beta}(b) \cdot f)(1, x, 0) = \psi(-b \det(x)) \cdot f(1, x, 0).$$

This formula will be useful in the last section.

8. Exceptional Theta Correspondences: $G \times G'$

Now we may study the restriction of the minimal representation Π to the dual pairs introduced earlier. In this section, we shall treat the family of dual pairs $G \times G'$ given in (23). For simplicity, we shall consider the case when $S = E_6$ and E_7 separately.

8.1. Restriction to $G \times G' \subset E_7$. Suppose first that S is of type E_7, so that Ω_Q is the set of rank 1 elements in $J = J(\mathbb{O})$. Consider the Schrodinger model for Π. On restricting Π to $Q_0 \times G$, we have the following formulae:

$$
\begin{cases}
(g \cdot f)(\alpha) = f(g^{-1} \cdot \alpha) & \text{for } g \in G; \\
(u(a) \cdot f)(\alpha) = \psi(tr(a \cdot \alpha)) \cdot f(\alpha) & \text{for } u(a) \in U_0; \\
(l \cdot f)(\alpha) = |\det(l)|^s \cdot f(l^{-1} \cdot \alpha) & \text{for } l \in L_0,
\end{cases}
$$

where s is a real number whose precise value will not be important to us here.

From our description of generic $L_0 \times G$-orbits given in §6.2, we deduce as in the derivation of (12) that as a $Q_0 \times G$-module,

$$
(25) \qquad \Pi \cong \bigoplus_{\chi \text{ generic}} \operatorname{Ind}_{U_0 \times \tilde{H}_\chi}^{Q_0 \times G} \chi \otimes 1 \cong \bigoplus_{\chi \text{ generic}} \operatorname{Ind}_{Z_\chi \cdot U_0}^{Q_0} L^2(H_\chi \backslash G).
$$

Here, G and $Z_\chi \cong \tilde{H}_\chi / H_\chi$ act on $L^2(H_\chi \backslash G)$ by right and left translation respectively, and U_0 acts by χ.

8.2. Abstract decomposition. On the other hand, there is an abstract direct integral decomposition

$$
\Pi = \int_{\widehat{G'}} \pi \otimes \Theta(\pi) \, d\nu_\Theta(\pi).
$$

Restricting to Q_0, we may write:

$$
\pi|_{Q_0} \cong \bigoplus_\chi \operatorname{Ind}_{Z_\chi \cdot U_0}^{Q_0} W_\chi(\pi)
$$

for some $Z_\chi \cdot U_0$-module $W_\chi(\pi)$ with U_0 acting via χ. Thus,

$$
(26) \qquad \Pi \cong \bigoplus_\chi \int_{\widehat{G'}} \operatorname{Ind}_{Z_\chi \cdot U_0}^{Q_0} W_\chi(\pi) \otimes \Theta(\pi) \, d\nu_\Theta(\pi).
$$

8.3. Comparison. Comparing (25) and (26), we deduce that there is an isomorphism of G-modules:

$$
(27) \qquad L^2(H_\chi \backslash G) \cong \int_{\widehat{G'}} W_\chi(\pi) \otimes \Theta(\pi) \, d\nu_\Theta(\pi).
$$

If G' is isogenous to a product of SL_2, the space $W_\chi(\pi) = Wh_\chi(\pi)$ has been determined in Theorem 5(3) and is at most 1-dimensional. If G' is $(Spin_3 \times Spin_5)/\Delta\mu_2$, Theorem 5(1, 2, 4), still gives some partial results on $W_\chi(\pi)$.

8.4. Mixed model. To explicate the measure $d\nu_\Theta(\pi)$, we consider the mixed model of Π restricted to $P \times G'$. Since

$$
N/Z_S = \hat{\mathfrak{k}} \oplus \mathcal{R}_0 \oplus \mathcal{R}_0' \oplus \hat{\mathfrak{k}} \subset N_S/Z_S.
$$

Under its adjoint action on $\mathcal{R} \oplus \hat{\mathfrak{k}}$, G' fixes $\mathcal{R}_0 \oplus \hat{\mathfrak{k}}$ pointwise, and its action on $\mathcal{R}_0^\perp$ is described in the following table.

G'	$\mathcal{R}_0$	$\mathcal{R}_0^{\perp}$
PGL_2	$J(\mathfrak{k})$	$adjoint^{\oplus 3}$
$SL_2^3/\Delta\mu_2$	$\mathfrak{k}^3$	$\oplus_{i=1}^3 std_i \otimes std_{i+1}^{\vee}$
$(Spin_3 \times Spin_5)/\Delta\mu_2$	$\mathfrak{k} \times \mathfrak{k}$	$(Spin \otimes Spin) \oplus (1 \otimes std)$

Thus as a representation of G', we have:

$$\Pi \cong L^2(\mathfrak{k}^{\times}) \otimes L^2(\mathcal{R}_0 \oplus \mathfrak{k}) \otimes L^2(\mathcal{R}_0^{\perp})$$

where G' acts only on $L^2(\mathcal{R}_0^{\perp})$ and the action is geometric. Thus, Π is weakly equivalent to $L^2(\mathcal{R}_0^{\perp})$ as a representation of G'. By our description of the G'-module $\mathcal{R}_0^{\perp}$, we have:

Lemma 16. *(i) If $G' = PGL_2$ or $SL_2^3/\Delta\mu_2$, the representation $L^2(\mathcal{R}_0^{\perp})$ (and hence Π) is weakly equivalent to the regular representation $L^2(G')$.*

(ii) If $G' = (Spin_3 \times Spin_5)/\Delta\mu_2$, the representation $L^2(\mathcal{R}_0^{\perp})$ (and hence Π) is weakly contained in the regular representation $L^2(G')$.

Proof. (i) When $G' = PGL_2$, this follows from Corollary 9. When $G' = SL_2^3/\Delta\mu_2$, the representation of G' on $E_A^{\perp}$ is the restriction of a representation of $\tilde{G}' = GL_2^3/\Delta\mathfrak{k}^{\times}$ (by the same formula). Now the action of $\tilde{G}'$ on $\mathcal{R}_0^{\perp}$ has finitely many open orbits with representatives $(1, 1, g) \in GL_2^3$ with g regular semisimple, and the stabilizer of such a representative is ΔT with T a maximal torus in PGL_2. Hence, as a representation of $\tilde{G}'$, $L^2(\mathcal{R}_0^{\perp})$ is weakly equivalent to

$$\bigoplus_T \mathrm{Ind}_{\Delta T}^{\tilde{G}'} \mathbb{C} \cong \bigoplus_T \mathrm{Ind}_{\Delta PGL_2}^{\tilde{G}'} L^2(T \backslash PGL_2)$$

as T runs over conjugacy classes of maximal tori in PGL_2. By Corollary 9 and the continuity of induction, we deduce that $L^2(\mathcal{R}_0^{\perp})$ is weakly equivalent to $L^2(\tilde{G}')$. Thus, on restriction to G', $L^2(E^{\perp})$ is weakly equivalent to $L^2(G')$, as desired.

(ii) We shall only give a sketch in this case. By considering the generic orbits of G' on $\mathcal{R}_0^{\perp}$ as in (i), one shows that $L^2(\mathcal{R}_0^{\perp})$ is weakly equivalent to the representation $L^2(\Delta Spin_3 \backslash Spin_3 \times Spin_5)$. One then checks that tempered matrix coefficients on $Spin_3 \times Spin_5$ are absolutely integrable on the subgroup $\Delta Spin_3$. Using the same argument as in [23, §6], one deduces that the spectral measure of $L^2(\Delta Spin_3 \backslash Spin_3 \times Spin_5)$ is absolutely continuous with respect to the Plancherel measure of G', whence the result.. $\square$

Concluding, we have:

Theorem 17. *There is an isomorphism of G-modules:*

$$L^2(H_\chi \backslash G) \cong \int_{\widehat{G'}} W_\chi(\pi) \otimes \Theta(\pi) \, d\mu_{G'}(\pi),$$

with $W_\chi(\pi)$ is some multiplicity space and $\mu_{G'}$ is the Plancherel measure. When $G' = PGL_2$ or $SL_2^3/\Delta\mu_2$, $W_\chi(\pi) = Wh_\chi(\pi)$ as given in Theorem 5(3).

In addition, as we discussed in §3.10, the smoooth analog of our argument in this section implies that when $G' = PGL_2$ or $SL_2^3/\Delta\mu_2$,

$$W_\chi(\pi) = Wh_\chi(\pi) \cong Hom_{H_\chi}(\Theta^{\infty}(\pi^{\infty}), \mathbb{C}) = Hom_{H_\chi}(\Theta(\pi)^{\infty}, \mathbb{C})$$

for $\mu_{G'}$-almost all π.

8.5. **Restriction to** $PGL_3 \times G_2$. We now treat the dual pair $PGL_3 \times G_2$ in $S = E_6$, which can be done by a similar analysis. In this case, $\Omega_Q \subset \mathbb{O}^2$. If we restrict the action of S to $Q_0 \times G_2$, we deduce by Lemma 15(ii) that as a representation of $Q \times G_2$,

$$\Pi \cong \mathrm{Ind}_{(P_{L_0} \times L \cdot [U,U])^0 \cdot U}^{Q_0 \times G_2} L^2(\hat{\mathfrak{k}})$$

where the action of $(P_{L_0} \times L \cdot [U,U])^0$ on $L^2(\hat{\mathfrak{k}})$ is given through the geometric action described in Lemma 15(ii) and the action of U_0 is by a nontrivial character fixed by P_{L_0}.

By using the Fourier transform on $L^2(\hat{\mathfrak{k}})$, we deduce that as a representation of $(P_{L_0} \times L \cdot [U,U])^0$,

$$L^2(\hat{\mathfrak{k}}) \cong \mathrm{Ind}_{U_{L_0} \times J}^{(P_{L_0} \times L \cdot [U,U])^0} \psi^{-1} \otimes \psi.$$

Hence, as a representation of $Q_0 \times G_2$

(28) $$\Pi \cong \mathrm{Ind}_{N_0}^{Q_0} \chi \otimes \mathrm{Ind}_J^{G_2} \psi$$

where $N_0 = U_{L_0} \cdot U_0$ is the unipotent radical of a Borel subgroup of PGL_3 and χ is a generic character of N_0.

On the other hand, we have abstractly

(29) $$\Pi \cong \int_{\widehat{PGL_3}} \pi|_{Q_0} \otimes \Theta(\pi) \, d\nu_\Theta(\pi).$$

We note that if π is tempered, then

$$\pi|_{Q_0} \cong \mathrm{Ind}_{N_0}^{Q_0} \chi,$$

in which case we deduce on comparing (28) and (29) that

(30) $$L^2((J, \psi)\backslash G_2) = \mathrm{Ind}_J^{G_2} \psi \cong \int_{\widehat{PGL_3}} \Theta(\pi) \, d\nu_\Theta(\pi).$$

For (30) to hold, we thus need to show that ν_Θ is absolutely continuous with respect to the Plancherel measure of PGL_3.

For this, we examine the mixed model of Π which is realized on $L^2(\hat{\mathfrak{k}}^\times \times J(\hat{\mathfrak{k}}^2) \times \hat{\mathfrak{k}})$. Noting that $J(\hat{\mathfrak{k}}^2) \cong \mathfrak{gl}_3$ as PGL_3-module [19], we deduce that as a representation of PGL_3, Π is weakly equivalent to the representation on $L^2(\mathfrak{sl}_3)$ associated to the adjoint action on $\mathfrak{sl}_3$. As in Corollary 9, we know that $L^2(\mathfrak{sl}_3)$ is weakly equivalent to $\bigoplus_T L^2(T\backslash PGL_3)$, with T running over conjugacy classes of maximal tori in PGL_3.

Using the same argument as in [23, §6], one can show that for each T, the spectral measure for $L^2(T\backslash PGL_3)$ is absolutely continuous with respect to the Plancherel measure of PGL_3, and hence so is the spectral measure of $L^2(\mathfrak{sl}_3)$; this justifies (30) and shows that

$$L^2((J, \psi)\backslash G_2) = \mathrm{Ind}_J^{G_2} \psi \cong \int_{\widehat{PGL_3}} W(\pi) \otimes \Theta(\pi) \, d\mu_{PGL_3}(\pi)$$

for some multiplicity space $W(\pi)$ of dimension ≤ 1.

It is natural to ask:

Question 18. *For which adjoint simple algebraic group G is the representation $L^2(\mathfrak{g})$ of G weakly equivalent to the regular representation $L^2(G)$?*

Corollary 9 verifies this conjecture for PGL_2. If the conjecture holds for PGL_3, one can then take $W(\pi)$ to be $\mathbb{C}$ for all π. In general, it is not difficult to show (using the argument in [23, §6] for example) that the support of the spectral measure of $L^2(\mathfrak{g})$ is contained in the tempered spectrum of G. We had initially conjectured that $L^2(\mathfrak{g})$ is weakly equivalent to $L^2(G)$. However, Kobayashi has

explained to us that for the adjoint group $G = PU(n, 1)$ (over $\mathbb{R}$), there is a family of holomorphic discrete series representations which does not occur in $L^2(\mathfrak{g})$. Since it is unclear what the correct statement is in general, we decide to formulate the above question.

9. Exceptional Theta Correspondence: $SL_3 \times Aut(\mathcal{R}, \det)$

Finally we come to the family of dual pairs $SL_3 \times Aut(\mathcal{R}, \det) \subset S = S_{\mathcal{R}}$ given by (22). What is interesting about this situation is that the group S may have no Siegel parabolic subgroup, so that the argument below is not the analog of that in the classical cases of §3. To simplify notation, we shall set $G = Aut(\mathcal{R}, \det)$. Note that in the case of F_4, S is the double cover of F_4 and the dual pair is $\tilde{SL}_3 \times G = \tilde{SL}_3 \times SL_3$.

Let $Q_0 = L_0 \cdot U_0 \subset SL_3$ be the maximal parabolic subgroup stabilizing the subspace $\mathfrak{k}e_1 + \mathfrak{k}e_2$, so that

$$L_0 \cong GL_2 \quad \text{and} \quad U_0 = u_{\beta_0 - \beta}(\mathfrak{k}) \times u_{\beta_0}(\mathfrak{k}).$$

Let χ be a generic character of U_0 trivial on $u_{\beta_0 - \beta}(\mathfrak{k})$. The stabilizer in L_0 of χ is a subgroup of the form $T_0 \ltimes U_{L_0}$ with $T_0 \cong \mathfrak{k}^\times$ contained in the diagonal torus and $U_{L_0} = u_{-\beta}(\mathfrak{k})$. On restricting the minimal representation Π to $Q_0 \times G$, we may write

$$\Pi \cong \mathrm{Ind}_{P_{L_0} U_0 \times G}^{Q_0 \times G} \Pi_\chi$$

for some representation Π_χ of $P_{L_0} U_0 \times G$ with U_0 acting by χ. Here, we have used the theorem of Howe-Moore which ensures that the trivial character of U_0 does not intervene.

Now we can describe the $P_{L_0} U_0 \times G$-module Π_χ using the mixed model of Π. Recall that this mixed model of Π is realized on $L^2(\mathfrak{k}^\times \times \mathcal{R} \times \mathfrak{k})$. Moreover, the action of $U_0 = u_{\beta_0 - \beta}(\mathfrak{k}) \times u_{\beta_0}(\mathfrak{k})$ in this model is:

$$\begin{cases} (u_{\beta_0}(z)f)(t, x, a) = \psi(tz) \cdot f(t, x, a) \\ (u_{\beta_0 - \beta}(y)f)(t, x, a) = \psi(ay) \cdot f(t, x, a). \end{cases}$$

As such, Π_χ is the representation obtained from Π by specializing (continuous) functions $f \in \Pi$ to the function $x \mapsto f(1, x, 0)$ of $\mathcal{R}$. Thus

$$\Pi_\chi = L^2(\mathcal{R})$$

where the action of $T_0 \times G$ is geometric, with T_0 acting by scaling on $\mathcal{R}$. Moreover, it follows by (24) that the action of $u_{-\beta}(b) \in U_{L_0}$ is:

$$(u_{-\beta}(b) \cdot f)(x) = \psi(-b \cdot \det(x)) \cdot f(x).$$

Now the set $\{x \in \mathcal{R} : \det(x) \neq 0\}$ is open dense and by Proposition 11(i), it is the union of finitely many generic orbits of $T_0 \times G$ indexed by $\mathfrak{k}^\times / (\mathfrak{k}^\times)^3$. For each $a \in \mathfrak{k}^\times / (\mathfrak{k}^\times)^3$, let H_a be the corresponding stabilizer group whose type is described in Table 4 in §4.2. Then

$$\Pi \cong \bigoplus_a \mathrm{Ind}_{N_0 \times H_a}^{Q_0 \times G} \chi_a \otimes \mathbb{C} \cong \mathrm{Ind}_{N_0}^{Q_0} \chi_a \otimes L^2(H_a \backslash G).$$

On the other hand, one has abstractly

$$\Pi \cong \int_{\widehat{SL_3}} \pi|_{Q_0} \otimes \Theta(\pi) \, d\nu_\theta(\pi).$$

Now we note:

Lemma 19. *As a representation of SL_3, Π is weakly equivalent to $L^2(SL_3)$.*

Proof. If S is of type E, the group SL_3 is contained in a conjugate of the Heisenberg parabolic subgroup P_S. Indeed, after an appropriate conjugation, we may assume that

$$SL_3 \subset Aut(J(\mathfrak{k}^2), \det) = SL_3 \times_{\mu_3} SL_3 \subset \mathrm{Aut}(J(B), \det),$$

where $B = \mathfrak{k}^2$, $M_2(\mathfrak{k})$ or the split octonion algebra $\mathbb{O}$ in the respective case. From the description of the mixed model, one sees that Π is nearly equivalent to the representation of SL_3 on $L^2(J(B)) = L^2(J(\mathfrak{k}^2)) \otimes L^2(J(\mathfrak{k}^2)^\perp)$. Since $J(\mathfrak{k}^2) \cong M_3(\mathfrak{k})$ with SL_3 acting by left multiplication, we see that $J(\mathfrak{k}^2)$ is weakly equivalent to the regular representation of SL_3. This implies that Π is weakly equivalent to the regular representation of SL_3.

The case when $S = F_4$ is a bit more intricate; we omit the details here. $\qquad\square$

Thus $\nu_\theta = \mu_{SL_3}$ and every π in the support of ν_θ is tempered, so that

$$\pi|_{Q_0} = \bigoplus_{a \in \mathfrak{k}^\times/(\mathfrak{k}^\times)^3} Wh_{\chi_a}(\pi) \otimes \mathrm{Ind}_{N_0}^{Q_0} \chi_a.$$

Comparing, we see that

$$L^2(H_a\backslash G) \cong \int_{\widehat{SL_3}} Wh_{\chi_a}(\pi) \otimes \Theta(\pi) d\mu_{SL_3}(\pi),$$

as desired.

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