Algebraic Topology

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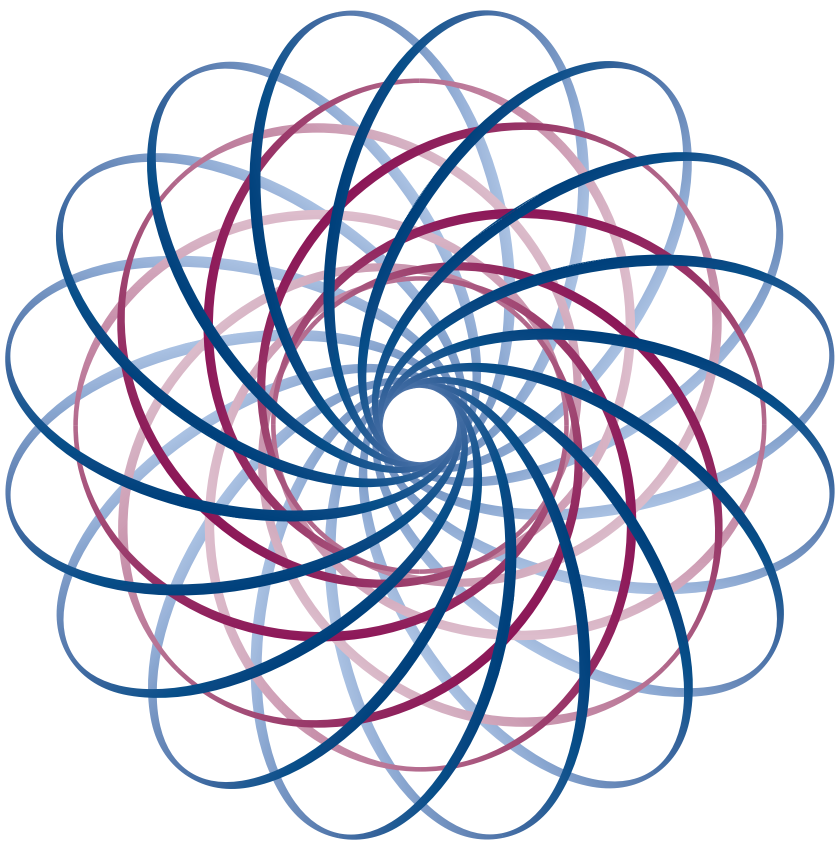

Table of Contents

Preface

This book was written to be a readable introduction to algebraic topology with rather broad coverage of the subject. The viewpoint is quite classical in spirit, and stays well within the confines of pure algebraic topology. In a sense, the book could have been written thirty or forty years ago since virtually everything in it is at least that old. However, the passage of the intervening years has helped clarify what are the most important results and techniques. For example, CW complexes have proved over time to be the most natural class of spaces for algebraic topology, so they are emphasized here much more than in the books of an earlier generation. This emphasis also illustrates the book's general slant towards geometric, rather than algebraic, aspects of the subject. The geometry of algebraic topology is so pretty, it would seem a pity to slight it and to miss all the intuition it provides.

At the elementary level, algebraic topology separates naturally into the two broad channels of homology and homotopy. This material is here divided into four chapters, roughly according to increasing sophistication, with homotopy split between Chapters 1 and 4, and homology and its mirror variant cohomology in Chapters 2 and 3. These four chapters do not have to be read in this order, however. One could begin with homology and perhaps continue with cohomology before turning to homotopy. In the other direction, one could postpone homology and cohomology until after parts of Chapter 4. If this latter strategy is pushed to its natural limit, homology and cohomology can be developed just as branches of homotopy theory. Appealing as this approach is from a strictly logical point of view, it places more demands on the reader, and since readability is one of the first priorities of the book, this homotopic interpretation of homology and cohomology is described only after the latter theories have been developed independently of homotopy theory.

Preceding the four main chapters there is a preliminary Chapter 0 introducing some of the basic geometric concepts and constructions that play a central role in both the homological and homotopical sides of the subject. This can either be read before the other chapters or skipped and referred back to later for specific topics as they become needed in the subsequent chapters.

Each of the four main chapters concludes with a selection of additional topics that the reader can sample at will, independent of the basic core of the book contained in the earlier parts of the chapters. Many of these extra topics are in fact rather important in the overall scheme of algebraic topology, though they might not fit into the time constraints of a first course. Altogether, these additional topics amount to nearly half the book, and they are included here both to make the book more comprehensive and to give the reader who takes the time to delve into them a more substantial sample of the true richness and beauty of the subject.

There is also an Appendix dealing mainly with a number of matters of a point-set topological nature that arise in algebraic topology. Since this is a textbook on algebraic topology, details involving point-set topology are often treated lightly or skipped entirely in the body of the text.

Not included in this book is the important but somewhat more sophisticated topic of spectral sequences. It was very tempting to include something about this marvelous tool here, but spectral sequences are such a big topic that it seemed best to start with them afresh in a new volume. This is tentatively titled 'Spectral Sequences in Algebraic Topology' and is referred to herein as [SSAT]. There is also a third book in progress, on vector bundles, characteristic classes, and K–theory, which will be largely independent of [SSAT] and also of much of the present book. This is referred to as [VBKT], its provisional title being 'Vector Bundles and K–Theory'.

In terms of prerequisites, the present book assumes the reader has some familiarity with the content of the standard undergraduate courses in algebra and point-set topology. In particular, the reader should know about quotient spaces, or identification spaces as they are sometimes called, which are quite important for algebraic topology. Good sources for this concept are the textbooks [Armstrong 1983] and [Jänich 1984] listed in the Bibliography.

A book such as this one, whose aim is to present classical material from a rather classical viewpoint, is not the place to indulge in wild innovation. There is, however, one small novelty in the exposition that may be worth commenting upon, even though in the book as a whole it plays a relatively minor role. This is the use of what we call Δ-complexes, which are a mild generalization of the classical notion of a simplicial complex. The idea is to decompose a space into simplices allowing different faces of a simplex to coincide and dropping the requirement that simplices are uniquely determined by their vertices. For example, if one takes the standard picture of the torus as a square with opposite edges identified and divides the square into two triangles by cutting along a diagonal, then the result is a Δ-complex structure on the torus having 2 triangles, 3 edges, and 1 vertex. By contrast, a simplicial complex structure on the torus must have at least 14 triangles, 21 edges, and 7 vertices. So Δ-complexes provide a significant improvement in efficiency, which is nice from a pedagogical viewpoint since it simplifies calculations in examples. A more fundamental reason for considering Δ-complexes is that they seem to be very natural objects from the viewpoint of algebraic topology. They are the natural domain of definition for simplicial homology, and a number of standard constructions produce Δ-complexes rather than simplicial complexes. Historically, Δ-complexes were first introduced by

Eilenberg and Zilber in 1950 under the name of semisimplicial complexes. Soon after this, additional structure in the form of certain 'degeneracy maps' was introduced, leading to a very useful class of objects that came to be called simplicial sets. The semisimplicial complexes of Eilenberg and Zilber then became 'semisimplicial sets', but in this book we have chosen to use the shorter term 'Δ-complex'.

This book will remain available online in electronic form after it has been printed in the traditional fashion. The web address is

http://www.math.cornell.edu/˜hatcher

One can also find here the parts of the other two books in the sequence that are currently available. Although the present book has gone through countless revisions, including the correction of many small errors both typographical and mathematical found by careful readers of earlier versions, it is inevitable that some errors remain, so the web page includes a list of corrections to the printed version. With the electronic version of the book it will be possible not only to incorporate corrections but also to make more substantial revisions and additions. Readers are encouraged to send comments and suggestions as well as corrections to the email address posted on the web page.

Note on the 2015 reprinting. A large number of corrections are included in this reprinting. In addition there are two places in the book where the material was rearranged to an extent requiring renumbering of theorems, etc. In §3.2 starting on page 210 the renumbering is the following:

old	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21
new	3.16	3.19	3.14	3.11	3.13	3.15	3.20	3.16	3.17	3.21	3.18

And in §4.1 the following renumbering occurs in pages 352–355:

old	4.13	4.14	4.15	4.16	4.17
new	4.17	4.13	4.14	4.15	4.16

Standard Notations

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$: the integers, rationals, reals, complexes, quaternions, and octonions.

$\mathbb{Z}_n$: the integers mod n.

$\mathbb{R}^n$: n-dimensional Euclidean space.

$\mathbb{C}^n$: complex n-space.
In particular, $\mathbb{R}^0 = \{0\} = \mathbb{C}^0$, zero-dimensional vector spaces.

$I = [0,1]$: the unit interval.

S^n : the unit sphere in $\mathbb{R}^{n+1}$, all points of distance 1 from the origin.

D^n : the unit disk or ball in $\mathbb{R}^n$, all points of distance ≤ 1 from the origin.

$\partial D^n = S^{n-1}$: the boundary of the n-disk.

e^n : an n-cell, homeomorphic to the open n-disk $D^n - \partial D^n$.
In particular, D^0 and e^0 consist of a single point since $\mathbb{R}^0 = \{0\}$.
But S^0 consists of two points since it is ∂D^1.

$\mathbb{1}$: the identity function from a set to itself.

$\coprod$: disjoint union of sets or spaces.

$\times$, $\prod$: product of sets, groups, or spaces.

$\approx$: isomorphism.

$A \subset B$ or $B \supset A$: set-theoretic containment, not necessarily proper.

$A \hookrightarrow B$: the inclusion map $A \to B$ when $A \subset B$.

$A - B$: set-theoretic difference, all points in A that are not in B.

iff : if and only if.

There are also a few notations used in this book that are not completely standard. The union of a set X with a family of sets Y_i, with i ranging over some index set, is usually written simply as $X \cup_i Y_i$ rather than something more elaborate such as $X \cup \left(\bigcup_i Y_i\right)$. Intersections and other similar operations are treated in the same way.

Definitions of mathematical terms are given within paragraphs of text, rather than displayed separately like theorems. These definitions are indicated by the use of **boldface type** for the more important terms, with italics being used for less important or less formal definitions as well as for simple emphasis as in standard written prose. Terms defined using boldface appear in the Index, with the page number where the definition occurs listed first.