In Proposition 3.22 of the first edition of the book the ring structure in $H^*(J(S^n);\mathbb{Z})$ was computed only for even n, but the calculation for odd n is not much harder so here is a revised version of the proposition that includes both cases.

Proposition 3.22. *For $n > 0$, $H^*(J(S^n);\mathbb{Z})$ consists of a $\mathbb{Z}$ in each dimension a multiple of n. If n is even, the i^{th} power of a generator of $H^n(J(S^n);\mathbb{Z})$ is $i!$ times a generator of $H^{in}(J(S^n);\mathbb{Z})$, for each $i \geq 1$. When n is odd, $H^*(J(S^n);\mathbb{Z})$ is isomorphic as a graded ring to $H^*(S^n;\mathbb{Z}) \otimes H^*(J(S^{2n});\mathbb{Z})$.*

It follows that for n even, $H^*(J(S^n);\mathbb{Z})$ can be identified with the subring of the polynomial ring $\mathbb{Q}[x]$ additively generated by the monomials $x^i/i!$. This subring is called a **divided polynomial algebra** and is denoted $\Gamma_{\mathbb{Z}}[x]$. Thus $H^*(J(S^n);\mathbb{Z})$ is isomorphic to $\Gamma_{\mathbb{Z}}[x]$ when n is even and to $\Lambda_{\mathbb{Z}}[x] \otimes \Gamma_{\mathbb{Z}}[y]$ when n is odd.

Proof: Giving S^n its usual CW structure, the resulting CW structure on $J(S^n)$ consists of exactly one cell in each dimension a multiple of n. If $n > 1$ we deduce immediately from cellular cohomology that $H^*(J(S^n);\mathbb{Z})$ consists exactly of $\mathbb{Z}$'s in dimensions a multiple of n. For an alternative argument that works also when $n = 1$, consider the quotient map $q:(S^n)^m \to J_m(S^n)$. This carries each cell of $(S^n)^m$ homeomorphically onto a cell of $J_m(S^n)$. In particular q is a cellular map, taking k-skeleton to k-skeleton for each k, so q induces a chain map of cellular chain complexes. This chain map is surjective since each cell of $J_m(S^n)$ is the homeomorphic image of a cell of $(S^n)^m$. Hence the cellular boundary maps for $J_m(S^n)$ will be trivial if they are trivial for $(S^n)^m$, as indeed they are since $H^*((S^n)^m;\mathbb{Z})$ is free with basis in one-to-one correspondence with the cells, by Theorem 3.16.

We can compute cup products in $H^*(J_m(S^n);\mathbb{Z})$ by computing their images under q^*. Let x_k denote the generator of $H^{kn}(J_m(S^n);\mathbb{Z})$ dual to the kn-cell, represented by the cellular cocycle assigning the value 1 to the kn-cell. Since q identifies all the n-cells of $(S^n)^m$ to form the n-cell of $J_m(S^n)$, we see from cellular cohomology that $q^*(x_1)$ is the sum $\alpha_1 + \cdots + \alpha_m$ of the generators of $H^n((S^n)^m;\mathbb{Z})$ dual to the n-cells of $(S^n)^m$. By the same reasoning we have $q^*(x_k) = \sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k}$.

If n is even, the cup product structure in $H^*((S^n)^m;\mathbb{Z})$ is strictly commutative and $H^*((S^n)^m;\mathbb{Z}) \approx \mathbb{Z}[\alpha_1,\cdots,\alpha_m]/(\alpha_1^2,\cdots,\alpha_m^2)$. Then we have

$$q^*(x_1^m) = (\alpha_1 + \cdots + \alpha_m)^m = m!\alpha_1 \cdots \alpha_m = m!q^*(x_m)$$

Since q^* is an isomorphism on H^{mn} this implies $x_1^m = m!x_m$ in $H^{mn}(J_m(S^n);\mathbb{Z})$. The inclusion $J_m(S^n) \hookrightarrow J(S^n)$ induces isomorphisms on H^i for $i \leq mn$ so we have $x_1^m = m!x_m$ in $H^*(J(S^n);\mathbb{Z})$ as well, where x_1 and x_m are interpreted now as elements of $H^*(J(S^n);\mathbb{Z})$.

When n is odd we have $x_1^2 = 0$ by commutativity, and it will suffice to prove the following two formulas:

(a) $x_1x_{2m} = x_{2m+1}$ in $H^*(J_{2m+1}(S^n);\mathbb{Z})$.

(b) $x_2x_{2m-2} = mx_{2m}$ in $H^*(J_{2m}(S^n);\mathbb{Z})$.

For (a) we apply q^* and compute in the exterior algebra $\Lambda_{\mathbb{Z}}[\alpha_1,\cdots,\alpha_{2m+1}]$:

$$\begin{aligned} q^*(x_1x_{2m}) &= \Big(\sum_i \alpha_i\Big)\Big(\sum_i \alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_{2m+1}\Big) \\ &= \sum_i \alpha_i\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_{2m+1} = \sum_i (-1)^{i-1}\alpha_1\cdots\alpha_{2m+1} \end{aligned}$$

The coefficients in this last summation are $+1,-1,\cdots,+1$, so their sum is $+1$ and (a) follows. For (b) we have

$$\begin{aligned} q^*(x_2x_{2m-2}) &= \Big(\sum_{i_1<i_2} \alpha_{i_1}\alpha_{i_2}\Big)\Big(\sum_{i_1<i_2} \alpha_1\cdots\widehat{\alpha}_{i_1}\cdots\widehat{\alpha}_{i_2}\cdots\alpha_{2m}\Big) \\ &= \sum_{i_1<i_2} \alpha_{i_1}\alpha_{i_2}\alpha_1\cdots\widehat{\alpha}_{i_1}\cdots\widehat{\alpha}_{i_2}\cdots\alpha_{2m} = \sum_{i_1<i_2} (-1)^{i_1-1}(-1)^{i_2-2}\alpha_1\cdots\alpha_{2m} \end{aligned}$$

The terms in the coefficient $\sum_{i_1<i_2}(-1)^{i_1-1}(-1)^{i_2-2}$ for a fixed i_1 have i_2 varying from i_1+1 to $2m$. These terms are $+1,-1,\cdots$ and there are $2m-i_1$ of them, so their sum is 0 if i_1 is even and 1 if i_1 is odd. Now letting i_1 vary, it takes on the odd values $1,3,\cdots,2m-1$, so the whole summation reduces to m 1's and we have the desired relation $x_2x_{2m-2} = mx_{2m}$. $\square$