Correction to *Algebraic Topology* by Allen Hatcher

The following corrects the last two paragraphs on page 335, Poincaré duality with local coefficients.

Cup and cap product work easily with local coefficients in a bundle of rings, the latter concept being defined in the obvious way. The cap product can be used to give a version of Poincaré duality for a closed n-manifold M using coefficients in a bundle of rings E under the same assumption as with ordinary coefficients that there exists a fundamental class $[M] \in H_n(M;E)$ restricting to a generator of $H_n(M, M - \{x\}; E)$ for all $x \in M$. By excision the latter group is isomorphic to the fiber ring R of E. The same proof as for ordinary coefficients then shows that $[M] \frown : H^k(M;E) \to H_{n-k}(M;E)$ is an isomorphism for all k.

Taking R to be one of the standard rings $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_p$ does not give anything new since the only ring automorphism these rings have is the identity, so the bundle of rings E must be the product $M \times R$. To get something more interesting, suppose we take R to be the ring $\mathbb{Z}[i]$ of Gaussian integers, the complex numbers $a + bi$ with $a, b \in \mathbb{Z}$. This has complex conjugation $a + bi \mapsto a - bi$ as a ring isomorphism. If M is nonorientable and connected we can use the homomorphism $\omega : \pi_1(M) \to \{\pm 1\}$ that defines the bundle of groups $M_{\mathbb{Z}}$ to build a bundle of rings E corresponding to the action of $\pi_1(M)$ on $\mathbb{Z}[i]$ given by $\gamma(a + bi) = a + \omega(\gamma) bi$. The homology and cohomology groups of M with coefficients in E depend only on the additive structure of $\mathbb{Z}[i]$ so they split as the direct sum of their real and imaginary parts, which are just the homology or cohomology groups with ordinary coefficients $\mathbb{Z}$ and twisted coefficients $\widetilde{\mathbb{Z}}$, respectively. The fundamental class in $H_n(M;\widetilde{\mathbb{Z}})$ constructed in Example 3H.3 can be viewed as a pure imaginary fundamental class $[M] \in H_n(M;E)$. Since cap product with $[M]$ interchanges real and imaginary parts, we obtain:

Theorem 3H.6. *If M is a nonorientable closed connected n-manifold then cap product with the pure imaginary fundamental class $[M]$ gives isomorphisms $H^k(M;\mathbb{Z}) \approx H_{n-k}(M;\widetilde{\mathbb{Z}})$ and $H^k(M;\widetilde{\mathbb{Z}}) \approx H_{n-k}(M;\mathbb{Z})$.* ☐

More generally this holds with $\mathbb{Z}$ replaced by other rings such as $\mathbb{Q}$ or $\mathbb{Z}_p$. There is also a version for noncompact manifolds using cohomology with compact supports.