The Kirby Torus Trick for Surfaces

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In the late 1960s the breakthrough that led to the Kirby–Siebenmann classification of PL structures on high-dimensional topological manifolds was the Kirby torus trick, appearing first in the paper [K] with a later exposition in Section I.3 of [KS]. It still seems quite amazing that a construction this simple could so easily convert difficult topological problems into much more manageable ones in the smooth or PL category.

In this paper we apply a scaled-down version of the torus trick to prove the existence and uniqueness of PL structures on surfaces, and in fact we take the small extra step to work with smoothings instead of triangulations. The inspiration for this was a 1974 paper by A.J.S.Hamilton [H] whose aim was to use the torus trick to prove the corresponding results for 3-manifolds, due originally to Moise. (It is not clear that the Hamilton paper achieved its goal due to the omission of a key argument which is left to the reader with the phrase "it is not difficult to see that ..." on page 67.)

Throughout the paper we take the term "surface" to mean a 2-dimensional manifold, possibly with boundary. We assume manifolds are Hausdorff and are covered by a countable number of coordinate charts.

Theorem A. *Every topological surface has a smooth structure.*

Theorem B. *Every homeomorphism between smooth surfaces is isotopic to a diffeomorphism.*

The latter result implies that smooth structures are unique not just up to diffeomorphism, but up to isotopy, by applying the theorem to the identity map between two different smooth structures on the same surface.

The proof of Theorem A will show that the smooth structure can be chosen to agree with a given smooth structure on a specified open set. Similarly, Theorem B can be refined to the statement that if a homeomorphism is already a diffeomorphism near the boundary of a surface, the isotopy can be chosen to be fixed in a neighborhood of the boundary.

The special feature of the proofs of Theorems A and B using the torus trick is that almost no point set topology is needed. This is replaced instead by a few basic facts about

smooth surfaces whose proofs use only smooth techniques. In particular the topological Schönflies theorem, which is an ingredient for most other proofs of Theorems A and B, is not needed here. An exception is the proof by Moise in [M] where the existence and uniqueness of PL structures on surfaces is proved before the Schönflies theorem.

The first proof that surfaces can be triangulated is generally attributed to Radó in 1925 [R]. That proof used the Schönflies theorem without explicitly saying so, but the later exposition of Radó's proof in [AS] makes the dependence clear. The fact that homeomorphisms of surfaces are isotopic to PL homeomorphisms (and hence to diffeomorphisms) was proved by Epstein in [E], although at least for closed orientable surfaces this can be derived from results of Baer in 1928 [B] on mapping class groups.

1. Reduction to Handle Smoothing.

Theorems A and B will be easy to deduce from a result about smoothing handles in surfaces. An i-handle in an n-manifold is usually taken to be a product $D^i \times D^{n-i}$ of closed disks but it will be more convenient here to use a product $D^i \times \mathbb{R}^{n-i}$ instead. For handles in surfaces there are three cases, for 0-handles, 1-handles, and 2-handles, and the proofs that they can be smoothed are somewhat different in each case, with the torus trick being needed only for 0-handles. The three cases could be combined into a single statement, but here are the separate statements.

Handle Smoothing Theorem. *Let S be a smooth surface. Then*:
(0) *A topological embedding $\mathbb{R}^2 \to S$ can be isotoped to be a smooth embedding in a neighborhood of the origin, staying fixed outside a larger neighborhood of the origin.*
(1) *A topological embedding $D^1 \times \mathbb{R} \to S$ which is a smooth embedding near $\partial D^1 \times \mathbb{R}$ can be isotoped to be a smooth embedding in a neighborhood of $D^1 \times 0$, staying fixed outside a larger neighborhood of $D^1 \times 0$ and near $\partial D^1 \times \mathbb{R}$.*
(2) *A topological embedding $D^2 \to S$ which is a smooth embedding in a neighborhood of ∂D^2 can be isotoped to be a smooth embedding on all of D^2, staying fixed in a smaller neighborhood of ∂D^2.*

It should be remembered that a smooth embedding is more than just a topological embedding which is a smooth map, since the differential must also be nonsingular at each point.

Assuming the Handle Smoothing Theorem, let us now deduce Theorems A and B. The proofs will use a few standard results about smooth surfaces, and these will be stated just as facts, with proofs given (or sketched) in the last section of the paper for those who

would like to see them. The same procedure will be followed later when we prove the Handle Smoothing Theorem.

Proof of Theorem A. For a surface S without boundary, choose a system of coordinate charts $h_i : \mathbb{R}^2 \to S$, $i = 1, 2, \cdots$. Let V_i be the image $h_i(\mathbb{R}^2)$, an open set in S. Each V_i has a smooth structure induced by h_i from the standard smooth structure on $\mathbb{R}^2$. The problem is to make these smooth structures agree on the overlaps of different V_i's.

Let U_n be the union $V_1 \cup \cdots \cup V_n$ and assume by induction on n that we have already modified $h_1, \cdots, h_n$ by isotopy to give a well-defined smooth structure on U_n. The induction starts with the case $n = 1$ where no modification of h_1 is needed for it to give a smooth structure on $U_1 = V_1$. The induction step will be to isotope h_{n+1} so that it restricts to a diffeomorphism from the open set $W = h_{n+1}^{-1}(U_n \cap V_{n+1})$ in $\mathbb{R}^2$ onto $U_n \cap V_{n+1}$ where $U_n \cap V_{n+1}$ has the smooth structure obtained by restriction of the smooth structure constructed inductively on U_n.

Fact 1. An open set $W \subset \mathbb{R}^2$ can be triangulated so that the size of the simplices approaches 0 at the frontier of W.

Having such a triangulation, the idea is to isotope h_{n+1} to make it a diffeomorphism on $W = h_{n+1}^{-1}(U_n \cap V_{n+1})$ inductively over neighborhoods of the skeleta of the triangulation. First, we choose disjoint 0-handles about all the vertices, with the vertices located at 0 in the 0-handles $\mathbb{R}^2$. Applying 0-handle smoothing then gives an isotopy of h_{n+1} to a smooth embedding, still called h_{n+1}, of an open neighborhood N_0 of the 0-skeleton into U_n. Next, for each edge of the triangulation choose a 1-handle $D^1 \times \mathbb{R}$ with $D^1 \times 0$ contained in the edge and $\partial D^1 \times \mathbb{R}$ contained in N_0. We can assume all these 1-handles are disjoint. Applying 1-handle smoothing to all the 1-handles gives an isotopy of h_{n+1} so that it becomes a smooth embedding of an open neighborhood N_1 of the 1-skeleton. Finally, we apply 2-handle smoothing to isotope h_{n+1} further to be a smooth embedding on the 2-handles obtained by slightly shrinking each 2-simplex of the triangulation, with the boundary of the 2-handle in N_1. The resulting h_{n+1} is a homeomorphism whose restriction $W \to U_n \cap V_{n+1}$ is locally a diffeomorphism and hence globally a diffeomorphism. This finishes the induction step in the proof of Theorem A when ∂S is empty.

To treat the case that ∂S is nonempty we will use the fact that ∂S has a collar neighborhood in S. This is a general fact about topological manifolds, with an elementary proof due to Connelly [C] that uses a partition of unity to piece together local collars. (For compact manifolds this argument is given in Proposition 3.42 in [Hat] and the argument extends easily to the noncompact case.) An open collar neighborhood U_0 of ∂S has a smooth structure since 1-manifolds are smoothable. Then the inductive procedure above

extends this to a smooth structure on all of S using coordinate charts $h_i : \mathbb{R}^2 \to S$ covering the interior of S, with $U_n = U_0 \cup V_1 \cup \cdots \cup V_n$. $\square$

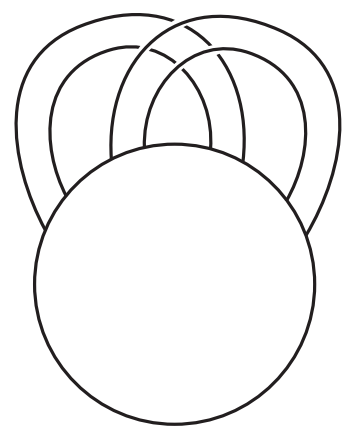
Proof of Theorem B. Let $f : S \to S'$ be a homeomorphism of smooth surfaces. We will isotope f to be a diffeomorphism by isotoping it on one handle at a time, with handles obtained from a triangulation of S using the following:

Fact 2. S has a smooth triangulation, with ∂S a subcomplex.

To apply this, suppose first that ∂S is empty. As in the proof of Theorem A we can do 0-handle smoothing to isotope f to be a diffeomorphism near all vertices, then apply 1-handle smoothing to isotope the new f to be a diffeomorphism near the 1-skeleton of the triangulation, and finally apply 2-handle smoothing to make the resulting f a diffeomorphism on all of S.

When ∂S is nonempty we can choose a smooth collar on ∂S. Using this, f can be isotoped to preserve the collar parameter in a smaller collar. The restriction of f to ∂S is isotopic to a diffeomorphism, and this isotopy can be extended to an isotopy of f, damped down to the constant isotopy as one moves into the collar away from ∂S. This gives a reduction to the case that f is already smooth on a neighborhood of ∂S. Then we can apply the smoothing procedure from the preceding paragraph to make f a diffeomorphism on the handles corresponding to simplices not contained in ∂S. $\square$

2. The Torus Trick: Smoothing 0-Handles.



This is the hardest case, where the torus trick is used. We view the torus T as the orbit space $\mathbb{R}^2/\mathbb{Z}^2$, with a basepoint 0 that is the image of $0 \in \mathbb{R}^2$. Deleting some other point $* \in T$ yields a punctured torus T'. We can immerse T' in $\mathbb{R}^2$, with $0 \in T'$ mapping to $0 \in \mathbb{R}^2$, by viewing T' as the interior of the surface obtained from a disk by attaching two 1-handles, then letting the immersion be an embedding of the disk, with the two 1-handles individually embedded so that their images cross in $\mathbb{R}^2$, as shown in the figure.

Let $h : \mathbb{R}^2 \to S$ be a topological embedding into the smooth surface S. Via h, the smooth structure on S induces a smooth structure $\mathcal{S}$ on $\mathbb{R}^2$. This is likely to be quite different from the usual smooth structure if h is far from being a smooth embedding. For example, smooth curves in S that are in the image of h pull back via h^{-1} to curves in

$\mathbb{R}^2$ that can be quite wild when looked at with the naked eye but are in fact smooth in the structure $\mathcal{S}$.

The immersion $T' \to \mathbb{R}^2$ pulls back the smooth structure $\mathcal{S}$ on $\mathbb{R}^2$ to a smooth structure on T' that we denote $T'_{\mathcal{S}}$.

Fact 3. There is a compact set in $T'_{\mathcal{S}}$ whose complement is diffeomorphic to $S^1 \times \mathbb{R}$.

This allows us to extend the smooth structure $T'_{\mathcal{S}}$ to a smooth structure $T_{\mathcal{S}}$ on T.

Fact 4. Every smooth structure on a torus $S^1 \times S^1$ is diffeomorphic to the standard smooth structure.

Thus there is a diffeomorphism $g : T_{\mathcal{S}} \to T$. Note that g, viewed just as a homeomorphism from T to T, is likely to be as complicated as the original h locally.

We can normalize g so that it takes 0 to 0 by composing with rotations in the S^1 factors of the target if necessary. We can then further normalize so that g induces the identity automorphism of $\pi_1(T, 0) = \mathbb{Z}^2$ by composing with a diffeomorphism in the target given by a suitable element of $GL_2(\mathbb{Z})$ acting on $\mathbb{R}^2/\mathbb{Z}^2$. After this normalization, g lifts to the universal covers as a diffeomorphism $\widetilde{g} : \mathbb{R}^2_{\mathcal{S}} \to \mathbb{R}^2$ fixing $\mathbb{Z}^2$, where the subscript $\mathcal{S}$ denotes the smooth structure lifted from $T_{\mathcal{S}}$. This lifting is periodic, satisfying $\widetilde{g}(x+m, y+n) = \widetilde{g}(x, y) + (m, n)$ for all $(x, y) \in \mathbb{R}^2$ and $(m, n) \in \mathbb{Z}^2$.

The key point of these constructions is that $\widetilde{g}$, as a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ is bounded, meaning that the distance $|\widetilde{g}(x, y) - (x, y)|$ is less than a fixed constant for all $(x, y) \in \mathbb{R}^2$. This is certainly true as (x, y) varies over the square I^2 by compactness, and then it holds over all of $\mathbb{R}^2$ by periodicity.

If we identify $\mathbb{R}^2$ with the interior of D^2 by a radial reparametrization, then $\widetilde{g}$ becomes a homeomorphism of the interior of D^2 that extends via the identity on ∂D^2 to a homeomorphism $G : D^2 \to D^2$, as a result of the boundedness condition. This can be seen by considering polar coordinates (r, θ), since as a disk in $\mathbb{R}^2$ of fixed radius moves out to infinity, the variation in the θ-coordinates of points in the disk approaches zero, and after the radial reparametrization the variation in the r-coordinates also approaches zero. We can choose the identification of $\mathbb{R}^2$ with the interior of D^2 to be the identity near 0, and then $G = \widetilde{g} = g$ near 0. Finally, we can extend $G : D^2 \to D^2$ to a homeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ that is the identity outside D^2.

By the Alexander trick G is isotopic to the identity. This isotopy can be obtained by replacing the unit disk D^2 in the preceding paragraph by disks of progressively smaller radius centered at 0, limiting to radius zero. Reversing the time parameter of this isotopy, we obtain an isotopy G_t with G_0 the identity and $G_1 = G$.

We can collect all the maps described above into one diagram:

$$\begin{array}{ccccccccc}
\mathbb{R}^2 & \longleftarrow & T' & \subset & T & \longleftarrow & \mathbb{R}^2 & \subset & D^2 & \subset & \mathbb{R}^2 \\
h\big\downarrow & & & & g\big\downarrow & & \widetilde{g}\big\downarrow & & G_t\big\downarrow & & G_t\big\downarrow \\
S & & & & T & \longleftarrow & \mathbb{R}^2 & \subset & D^2 & \subset & \mathbb{R}^2
\end{array}$$

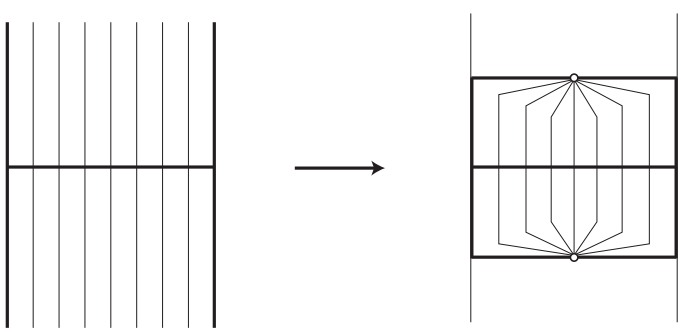
We check now that the desired isotopy of h is given by $h_t = hG_t^{-1}$. We have $h_0 = h$ since G_0 is the identity. Also h_t is stationary outside D^2 since G_t is the identity there. Since G_t fixes the origin for all t, we have $h_t(0) = h(0)$ for all t. Finally, to check that h_1 is a smooth embedding near 0 with respect to the standard smooth structure on $\mathbb{R}^2$ we have $h_1 = hG_1^{-1} = hG^{-1}$ with G^{-1} a diffeomorphism from the standard smooth structure to the smooth structure $\mathcal{S}$ near 0, and h carries the smooth structure $\mathcal{S}$ to the smooth structure that was given on S. $\square$

3. Smoothing 1-Handles and 2-Handles.

For smoothing a 1-handle we are given an embedding $h : D^1 \times \mathbb{R} \to S$ which is smooth near $\partial D^1 \times \mathbb{R}$. Via h the smooth structure on S pulls back to a smooth structure $\mathcal{S}$ on $D^1 \times \mathbb{R}$ which agrees with the standard structure near $\partial D^1 \times \mathbb{R}$.

Fact 5. Every smooth structure on $D^1 \times \mathbb{R}$ that is standard near $\partial D^1 \times \mathbb{R}$ is diffeomorphic to the standard structure via a diffeomorphism that is the identity near $\partial D^1 \times \mathbb{R}$.

Thus there is a diffeomorphism $g : (D^1 \times \mathbb{R})_{\mathcal{S}} \to D^1 \times \mathbb{R}$ which is the identity near $\partial D^1 \times \mathbb{R}$. Choose an embedding $e : D^1 \times \mathbb{R} \to D^1 \times \mathbb{R}$ which is the identity near $D^1 \times 0$ and which has image $(D^1 \times D^1) - (0 \times \partial D^1)$, as indicated in the figure below.



Via e, g defines a homeomorphism of the image of e, and this homeomorphism can be extended by the identity outside this image to yield a homeomorphism $G : D^1 \times \mathbb{R} \to D^1 \times \mathbb{R}$ that agrees with g near $D^1 \times 0$ and is the identity outside $D^1 \times D^1$ and near $\partial D^1 \times \mathbb{R}$. If we perform the Alexander isotopy on G, using the square $D^1 \times D^1$ this time instead of the disk D^2, we obtain, after reversing the time parameter, an isotopy $G_t : D^1 \times \mathbb{R} \to D^1 \times \mathbb{R}$

from the identity to G. This is stationary near $\partial D^1 \times \mathbb{R}$ and outside $D^1 \times D^1$.

$$\begin{array}{ccccc} S & \xleftarrow{h} & D^1 \times \mathbb{R} & \xrightarrow{g} & D^1 \times \mathbb{R} \\ & & e\big\downarrow & & \\ & & D^1 \times \mathbb{R} & \xrightarrow{G_t} & D^1 \times \mathbb{R} \end{array}$$

The desired isotopy of h is then $h_t = hG_t^{-1}$ since G_0 is the identity and $G_1 = G = g$ near $D^1 \times 0$ where $h_1 = hg^{-1}$ is the composition $D^1 \times \mathbb{R} \to (D^1 \times \mathbb{R})_S \to S$ of a diffeomorphism followed by a smooth embedding. $\square$

Finally we have the case of smoothing 2-handles. Here we start with an embedding $h : D^2 \to S$ which is smooth near ∂D^2. The smooth structure on S pulls back to a smooth structure $D^2_{\mathcal{S}}$ which is standard near ∂D^2.

Fact 6. Every smooth structure on D^2 that is standard near ∂D^2 is diffeomorphic to the standard structure via a diffeomorphism that is the identity near ∂D^2.

Thus there is a diffeomorphism $g : D^2_{\mathcal{S}} \to D^2$ which is the identity near ∂D^2. The Alexander trick yields an isotopy $g_t : D^2 \to D^2$ from the identity to g, and then an isotopy from h to a smooth embedding is given by $h_t = hg_t^{-1}$, fixing a neighborhood of ∂D^2. $\square$

4. Proofs of the Facts.

Fact 1. *An open set $W \subset \mathbb{R}^2$ can be triangulated so that the size of the simplices approaches 0 at the frontier of W.*

Proof. The horizontal and vertical lines in $\mathbb{R}^2$ through points in the integer lattice $\mathbb{Z}^2$ divide $\mathbb{R}^2$ into closed unit squares. To start, choose all such squares that are contained in the open set W. For the squares that are not contained in W, divide each of these into four squares with sides of length $1/2$ and take all the smaller squares that lie in W. Now repeat this process indefinitely, at each stage subdividing the squares of the preceding stage not contained in W into four subsquares and choosing the ones contained in W. The union of all the squares chosen by this iterative process is W since the distance from any point in W to $\mathbb{R}^2 - W$ is positive. Only finitely many of the squares touch a given square since the distance from the square to $\mathbb{R}^2 - W$ is positive. The chosen squares give a cellulation of W, and this can be subdivided to a triangulation by adding a new vertex at the center of each square and joining this vertex to the vertices in the boundary of the square. $\square$

Fact 2. *Every smooth surface S has a smooth triangulation with ∂S a subcomplex.*

Proof. Consider first the case that ∂S is empty. We will construct a smooth cellulation of S, with 2-cells that are polygons, glued together along their edges. Choose a Morse function $S \to \mathbb{R}$ that is proper (inverse images of compact sets are compact) and has all its critical points on distinct levels. Cutting S along noncritical levels separating all critical levels then gives rise to a decomposition of S into pieces that are diffeomorphic to disks, annuli, pairs of pants, or annuli with a 1-handle attached to one boundary circle so as to produce a nonorientable surface. A surface of this last type can be further cut along a circle in its interior to produce a pair of pants and a Möbius band. Thus S is decomposed into disks, annuli, pairs of pants, and Möbius bands, glued together along their boundary circles. From this a cellulation is easily obtained by inserting one vertex in each circle, then in each annulus connecting the two vertices in its boundary by an arc cutting the annulus into a square, in each pair of pants connecting the three vertices in its boundary by two arcs cutting the pair of pants into a polygon (a heptagon in fact), and in each Möbius band inserting an edge to cut it into a triangle. (Think of the Möbius band as a rectangle with two opposite edges identified, then cut the rectangle along a diagonal before identifying the opposite edges.) Having a smooth cellulation of S, we can then subdivide it to a smooth triangulation, although for the application of Fact 2 in Section 1 a smooth cellulation would suffice.

The extension of this construction to the case that ∂S is nonempty is similar, using a proper Morse function with all its critical points in the interior of S and with its restriction to ∂S a proper Morse function on ∂S. Details are left to the reader. $\square$

Fact 3. *If $T'_{\mathcal{S}}$ is a smooth structure on the punctured torus T' then there is a compact set in $T'_{\mathcal{S}}$ whose complement is diffeomorphic to $S^1 \times \mathbb{R}$.*

Proof. As in the proof of Fact 2, $T'_{\mathcal{S}}$ can be cut along a collection of disjoint smooth circles C_i to produce pieces P_j that are disks, annuli, pairs of pants, and Möbius bands, but the last of these cannot occur since $T'_{\mathcal{S}}$ is orientable. (Orientability can be defined purely topologically, either in terms of local homology groups or in terms of local fundamental groups, which are fundamental groups of neighborhoods of a point with the point deleted.) The pattern in which the pieces P_j are assembled to form $T'_{\mathcal{S}}$ is described by a graph G having one vertex for each P_j and one edge for each circle C_i. There is a quotient map $q : T'_{\mathcal{S}} \to G$ projecting a product neighborhood of each C_i onto its interval factor, which becomes the corresponding edge of G, and collapsing the complementary components of these annuli to points, the vertices of G. The induced homomorphism $q_* : \pi_1 T'_{\mathcal{S}} \to \pi_1 G$ is

surjective and in fact split since q has a section up to homotopy. Thus the free group $\pi_1 G$ is a quotient of $\pi_1 T'_{\mathcal{S}}$, so it is finitely generated. This implies that there is a finite subgraph $G_0 \subset G$ such that the closure of $G - G_0$ consists of a finite number of trees. Only one of these trees can be noncompact since $T'_{\mathcal{S}}$ has only one end. The one-endedness also implies that this noncompact tree consists of an infinite subtree homeomorphic to $[0, \infty)$ with finite subtrees attached to it. We can eliminate these finite subtrees by deleting the edges leading to vertices corresponding to disk pieces P_j, discarding the circles C_i corresponding to these edges. After this has been done the end of $T'_{\mathcal{S}}$ consists of an infinite sequence of annuli glued together to form an infinite cylinder. The glueings are smooth so the cylinder is diffeomorphic to a standard cylinder. $\square$

Fact 4. *Every smooth structure on a torus $S^1 \times S^1$ is diffeomorphic to the standard smooth structure.*

Proof. We could just quote the classification of smooth closed surfaces, or we can give an argument similar to the one for Fact 3. Let the given smooth structure on the torus be denoted $T_{\mathcal{S}}$ and choose a Morse function $T_{\mathcal{S}} \to \mathbb{R}$. The associated graph G is now a finite graph with $\pi_1 G$ a quotient of the abelian group $\pi_1 T_{\mathcal{S}}$, so $\pi_1 G$ is either $\mathbb{Z}$ or the trivial group. In the former case we can reduce G to a circle by eliminating vertices corresponding to disk pieces as before, and then we see that $T_{\mathcal{S}}$ consists of annuli glued together in a cyclic pattern, so $T_{\mathcal{S}}$ is diffeomorphic to a torus or a Klein bottle, but the latter is ruled out since $\pi_1 T_{\mathcal{S}}$ is abelian (or since $T_{\mathcal{S}}$ is orientable). In the other case that $\pi_1 G$ is trivial we can reduce G to a single edge, which would mean that $T_{\mathcal{S}}$ was a sphere, so this case cannot occur. $\square$

Fact 5. *Every smooth structure on $D^1 \times \mathbb{R}$ that is standard near $\partial D^1 \times \mathbb{R}$ is diffeomorphic to the standard structure via a diffeomorphism that is the identity near $\partial D^1 \times \mathbb{R}$.*

Proof. Let $(D^1 \times \mathbb{R})_{\mathcal{S}}$ be the given smooth structure on $D^1 \times \mathbb{R}$. The projection $D^1 \times \mathbb{R} \to \mathbb{R}$ can be perturbed to a proper Morse function $f : (D^1 \times \mathbb{R})_{\mathcal{S}} \to \mathbb{R}$ staying fixed near $\partial D^1 \times \mathbb{R}$ where it is already smooth without critical points. We can assume all the critical points of f lie in distinct levels. Noncritical levels consist of a single arc and finitely many circles. Cutting along these curves for a set of noncritical levels separating all the critical levels produces pieces that are disks, annuli, or pairs of pants as in the earlier cases, and there are now also two new kinds of pieces: rectangles, and rectangles with an open disk removed. The associated graph G is a tree since $\pi_1(D^1 \times \mathbb{R}) = 0$, and G has two ends since $(D^1 \times \mathbb{R})_{\mathcal{S}}$ has two ends. Thus G consists of a subgraph homeomorphic to $\mathbb{R}$ with finite trees attached to it. These finite trees can be eliminated as before, and f can be modified

to a proper Morse function without critical points, equal to the old f near $\partial D^1 \times \mathbb{R}$. The new f is then the second coordinate of a diffeomorphism $g : (D^1 \times \mathbb{R})_{\mathcal{S}} \to D^1 \times \mathbb{R}$ whose first coordinate is obtained by flowing up or down from $f^{-1}(0)$ along the integral curves of the gradient vector field of f. These curves are standard near $(\partial D^1 \times \mathbb{R})_{\mathcal{S}}$ so g can be taken to be the identity there. $\square$

Fact 6. *Every smooth structure on D^2 that is standard near ∂D^2 is diffeomorphic to the standard structure via a diffeomorphism that is the identity near ∂D^2.*

Proof. Here we extend the radial coordinate in D^2 near ∂D^2 to a Morse function $f : D^2_{\mathcal{S}} \to [0,1]$ with $f^{-1}(1) = \partial D^2_{\mathcal{S}}$ and all critical points in the interior of $D^2_{\mathcal{S}}$. The associated graph is a finite tree, and it can be simplified as before until it is a line segment, with a new f having a single critical point, a local minimum. Then f together with the flow lines of its gradient field can be used to construct a diffeomorphism $g : D^2_{\mathcal{S}} \to D^2$ which is the identity near $\partial D^2_{\mathcal{S}}$. $\square$

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