Topological Moduli Spaces of Knots

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Classical knot theory is concerned with isotopy classes of knots in the 3-sphere, in other words, path-components of the space $\mathcal{K}$ of all smooth submanifolds of S^3 diffeomorphic to the circle S^1. What can be said about the homotopy types of these various path-components? One would like to find, for the path-component $\mathcal{K}_K$ containing a given knot K, a small subspace $\mathcal{M}_K$ to which $\mathcal{K}_K$ deformation retracts, thus a minimal homotopic model or moduli space for $\mathcal{K}_K$. In this paper we describe a reasonable candidate for $\mathcal{M}_K$ and prove for many knots K that $\mathcal{K}_K$ does indeed have the homotopy type of the model $\mathcal{M}_K$. The proof would apply for all K provided that a certain well-known conjecture in 3-manifold theory is true, the conjecture that every free action of a finite cyclic group on S^3 is equivalent to a standard linear action.

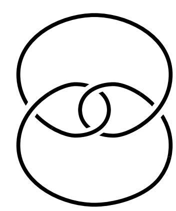
The model $\mathcal{M}_K$ takes a particularly simple form if K is either a torus knot or a hyperbolic knot. In these cases $\mathcal{M}_K$ is a single orbit of the action of $SO(4)$ on $\mathcal{K}_K$ by rotations of the ambient space S^3, namely an orbit of a "maximally symmetric" position for K, a position where the subgroup $G_K \subset SO(4)$ leaving K setwise invariant is as large as possible. The orbit is thus the coset space $SO(4)/G_K$. The assertion that $\mathcal{K}_K$ deformation retracts to $\mathcal{M}_K = SO(4)/G_K$ is then a sort of *homotopic rigidity* property of K.

This picture is consistent with the intuition that there should exist a nice physically meaningful energy function on $\mathcal{K}_K$ whose only critical points are global minima forming a copy of the manifold $\mathcal{M}_K$, such that the associated gradient flow gives a deformation retraction of $\mathcal{K}_K$ onto $\mathcal{M}_K$. One would expect the energy function to be invariant under rotations of S^3, so its critical point set would be a union of orbits and ideally just a single orbit. The flow should respect any symmetry the knot might have, so the minimum energy position for the knot should also be a maximally symmetric position.

Let us describe the models $\mathcal{M}_K$ in more detail in the cases of torus knots and hyperbolic knots. Consider first the case of the trivial knot. Its most symmetric position is clearly a great circle in S^3. The subgroup $G_K \subset SO(4)$ taking K to itself is then the index two subgroup of a standard $O(2)\times O(2)$ in $O(4)$ consisting of orientation-preserving isometries. It was shown in [H1] that $\mathcal{K}_K$ has the homotopy type of the orbit $\mathcal{M}_K = SO(4)/G_K$, which can be identified with the 4-dimensional Grassmann manifold of 2-planes through the origin in $\mathbb{R}^4$.

Consider next a nontrivial torus knot $K = K_{p,q}$ for relatively prime integers p and q, neither of which is ± 1. Regarding S^3 as the unit sphere in $\mathbb{C}^2$, the most symmetric position for K is as the set of points $(z^p, z^q)/\sqrt{2}$ with $|z| = 1$. The symmetry group then contains the unitary diagonal matrices with z^p and z^q as diagonal entries, forming a subgroup $S^1 \subset SO(4)$. There is also a rotational symmetry reversing the orientation of K, given by complex conjugation in each variable. Thus G_K contains a copy of $O(2)$, whose restriction to K is the usual action of $O(2)$ on S^1. Clearly G_K cannot be larger than this, otherwise K would be pointwise fixed by a nontrivial element of $SO(4)$, hence would be unknotted. We will show that $\mathcal{K}_K$ has the homotopy type of the orbit $SO(4)/G_K$, a closed 5-manifold.

Now let K be hyperbolic, so $S^3 - K$ has a unique complete hyperbolic structure, and let Γ_K be the finite group of orientation-preserving isometries of this hyperbolic structure. By the theorem of Gordon-Luecke [GL], elements of Γ_K must take meridians of K to meridians, so the action of Γ_K on $S^3 - K$ extends to an action on S^3. By the Smith conjecture [MB], no nontrivial elements of Γ_K fix K pointwise, so Γ_K is a group of diffeomorphisms of K, hence Γ_K must be cyclic or dihedral. Assuming the action of Γ_K on S^3 is equivalent to an action by elements of $SO(4)$, then we can isotope K to a symmetric position in which the action is by isometries of S^3, so we have an embedding $\Gamma_K \subset SO(4)$. We will show in this case that $\mathcal{K}_K$ has the homotopy type of $SO(4)/\Gamma_K$. The symmetry group G_K cannot be larger than Γ_K, as we will see, so in this symmetric position for K we have $G_K = \Gamma_K$, and $SO(4)/\Gamma_K$ is the orbit of K under the $SO(4)$ action.

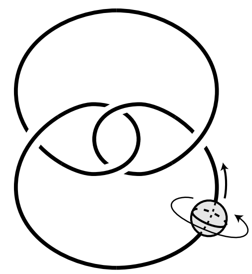


For example, if K is the figure eight knot then Γ_K is $Z_2 \times \mathbb{Z}_2$, generated by the 180 degree rotations about orthogonal axes in the picture at the right. The group $\pi_1 \mathcal{K}_K \approx \pi_1(SO(4)/\Gamma_K)$ has order eight, and is in fact the quaternion group, with the 360 degree rotations of K about its axes of symmetry lifting to generators of $\pi_1 SO(4)$.

The hypothesis that the action of Γ_K on S^3 is equivalent to an isometric action can be restated as the well-known conjecture that the orbifold S^3/Γ_K has a spherical structure, a special case of Thurston's geometrization conjecture for orbifolds. We call it *the linearization conjecture for* K. It is a theorem of Thurston that the geometrization conjecture is true for orbifolds which are not actually manifolds. So the linearization conjecture for K is true unless Γ_K is a cyclic group acting freely on S^3. It is also known to be true for free actions by a cyclic group of order two [L], a power of two [M], or three [MR]. Random knots have no symmetry, so the linearization conjecture holds vacuously for them. For knots that do have symmetries, the chances are that the symmetry group will be small, of order less than five, or will contain rotations about some fixed axis, so in these cases the linearization conjecture holds as well.

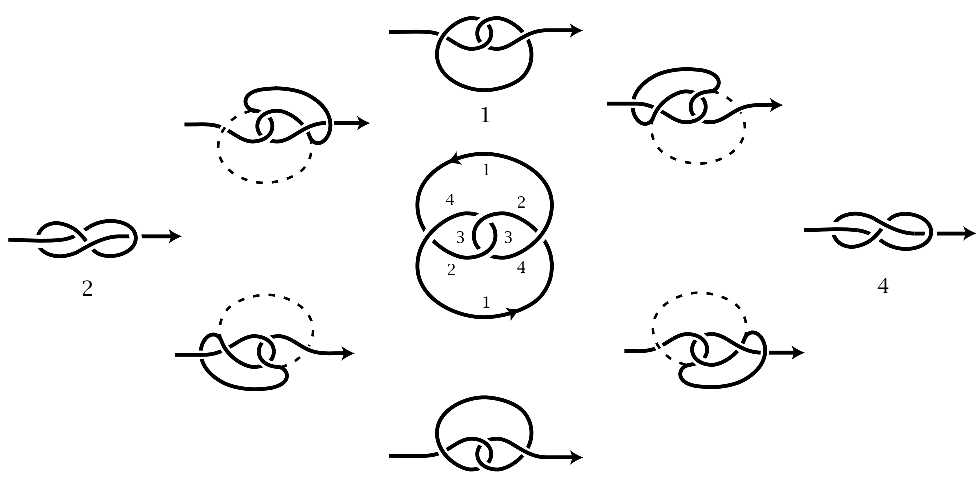
The unknown cases would seem to be quite rare.

We also analyze the space $\mathcal{A}$ of knotted arcs in a ball with fixed endpoints on the boundary sphere. We show that $\mathcal{A}_K$, the path-component of $\mathcal{A}$ corresponding to the knot K, is a $K(\pi,1)$ for every knot K. When K is trivial, $\mathcal{A}_K$ is contractible by [H1], and we show that $\mathcal{A}_K \simeq S^1$ when K is a nontrivial torus knot and $\mathcal{A}_K \simeq S^1 \times S^1$ when K is hyperbolic. Nice models for $\mathcal{A}_K$ in these cases can be described as follows. We can convert a knotted circle K into a knotted arc in a ball by placing a small bead B on a wire model of K, and then the part of K outside B becomes a knotted arc in the ball complementary to B in S^3. The endpoints of the arc are almost antipodal on the boundary of the ball, and after a small adjustment they can be taken to be exactly antipodal. As the bead moves around the knot K we obtain in this way a circle's worth of knotted arcs. We can also twirl a knotted arc in a ball about the axis through its two antipodal endpoints to get another circle's worth of knotted arcs. This corresponds to twirling the bead about its axis.



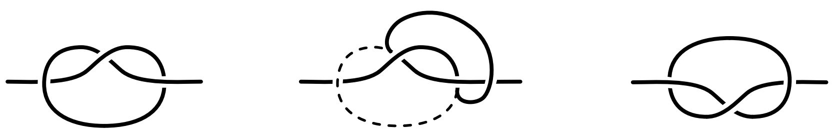
Thus we have a map $S^1 \times S^1 \to \mathcal{A}_K$. If K is a nontrivial torus knot in its standard position, the image of this map is just a circle in $\mathcal{A}_K$, and this circle is the model for $\mathcal{A}_K$. If K is a hyperbolic knot satisfying the linearization conjecture and K is in symmetric position, then the map $S^1 \times S^1 \to \mathcal{A}_K$ has image a torus, the map being a cyclic covering space of its image with deck transformations corresponding to the symmetries of K preserving an orientation of K. This torus in $\mathcal{A}_K$ is the model for $\mathcal{A}_K$ in this case.

As an example, consider the figure eight knot, in the symmetric position shown in the center of the following figure.



Placing the bead at a point of the knot in each of the eight arcs between successive crossings gives eight positions for a knotted arc, but by symmetry only four of these are distinct, indicated by the numbers 1-4 in the figure. Regarding the projection plane of the figure as a 2-sphere by adding a point at infinity, the four positions for the knotted arc are shown at the four corners of the figure. Going from position 1 to position 2, for example, the bead passes through an undercrossing, and this has the effect of dragging the overcrossing strand across the front of the knot so that it becomes an overcrossing at the other end of the knotted arc. This is shown in the second quadrant of the figure. Similarly, moving the bead from position 1 to position 4 has the effect shown in the first quadrant. The other two quadrants show the transitions from position 3 to positions 2 and 4. Thus as the bead goes halfway around the knot we obtain a loop of knotted arcs. Independently of this deformation we can also twirl the knotted arc about an axis through its two endpoints, to get the other S^1 factor of the model for $\mathcal{A}_K$.

The same procedure applied to the trefoil knot produces a loop of embedded arcs that is homotopic to the loop obtained by simply twirling the knotted arc about the axis through its two endpoints.



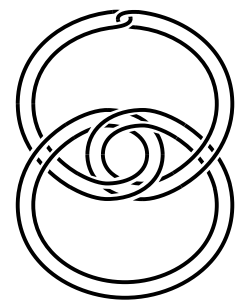
The same thing happens for all torus knots.

The proofs of these results for torus knots and hyperbolic knots are fairly short, and consist mainly of invoking a number of nontrivial theorems from 3-manifold theory.

Satellite Knots

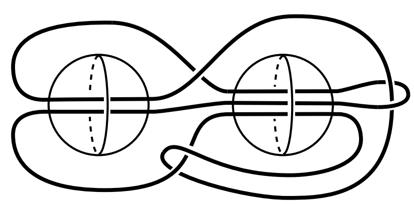
Knots which are not torus knots or hyperbolic knots are satellite knots, and for these the analysis of $\mathcal{K}_K$ becomes more complicated. In particular the homotopic rigidity property of torus knots and hyperbolic knots fails for satellite knots. Modulo the linearization conjecture again, we show that $\mathcal{K}_K$ has the homotopy type of a model $\mathcal{M}_K$ which is a finite-dimensional manifold of the form $(SO(4)\times X_K)/\Gamma_K$ where Γ_K is a compact group of "supersymmetries" of K and X_K is the product of a torus of some dimension and a number of configuration spaces C_n of ordered n-tuples of distinct points in $\mathbb{R}^2$. These configuration spaces occur only when the satellite structure of K involves nonprime knots. When they are present, $\pi_1\mathcal{K}_K$ involves braid groups, a phenomenon observed first in [G] in the case that K itself is nonprime. When there are no configuration spaces, $\mathcal{M}_K$ is a closed manifold, but in general there can be no closed manifold model for $\mathcal{K}_K$ since its homology groups may not satisfy Poincaré duality. The space X_K appearing in the definition of $\mathcal{M}_K$ is determined just by the general form of the satellite structure of K, while the group Γ_K is more delicate, depending strongly on the particular knots appearing in the satellite structure. The components of Γ_K are tori arising from cabling in the satellite structure, and $\pi_0\Gamma_K$ is the quotient of $\pi_0\mathrm{Diff}^+(S^3 \text{ rel } K)$ by the subgroup generated by Dehn twists along essential tori in $S^3 - K$. One could cancel the tori components of Γ_K with some circle factors of X_K and thereby make Γ_K a finite group in all cases, but from a conceptual viewpoint this does not seem a very natural thing to do.

As an example, consider the knot K shown in the figure, a Whitehead double of the figure eight knot. Thus K lies in a tubular neighborhood V of a figure eight knot. The solid torus V has a 2-parameter family of homeomorphisms given by longitudinal and meridional rotations of its circle and disk factors. Restricting these homeomorphisms to K we obtain a family of embeddings of K in V. This family is the space X_K, a torus $S^1 \times S^1$. We can also apply rotations of the ambient space S^3 to get a map $SO(4) \times X_K \to \mathcal{K}_K$. This is not injective, but is in fact eight-to-one, assuming that V has been chosen symmetrically. Namely, a 180 degree meridional rotation of V takes K to itself, and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ rotational symmetry group of the figure eight knot also acts on X_K. The group Γ_K in this case is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Factoring out its action on $SO(4) \times X_K$ we obtain an injection $(SO(4) \times X_K)/\Gamma_K \to \mathcal{K}_K$ which is a homotopy equivalence.



It is not hard to see how to iterate this construction in the case that the satellite structure for K consists of a nested sequence of tori, for example if K is built by iterated Whitehead doubles of a hyperbolic or torus knot. In general, however, the tori defining the satellite structure need not form a single nested sequence, and then a different construction is needed.

Here is an example to give the essential idea. Start with a knot K', which could be trivial, and choose a number of disjoint balls each of which intersects K' in two or more parallel unknotted arcs. We imagine these parallel arcs as passing through an unknotted tube going through the ball. Now tie each tube in a knot, say a hyperbolic knot or a turus knot, carrying along the strands of K' so that they now run in parallel through the knotted tube. The new knot K' is the knot K we are interested in. Let us view each knotted tube as a tubular neighborhood of a knotted arc. We have seen earlier how a knotted arc belongs to a family of knotted arcs parametrized by points of a torus $S^1 \times S^1$. Carrying the tube and the strands of K inside the tube along during the 2-parameter deformation, we obtain a family of embeddings of K. Thus we have a torus factor of X_K for each knotted tube. As in the previous cases, symmetries must be factored out to obtain the model $(SO(4) \times X_K)/\Gamma_K$.



1. Homotopically Rigid Knots

For a knot $K \subset S^3$ the action of the group $\mathrm{Diff}^+(S^3)$ of orientation-preserving diffeomorphisms of S^3 on $\mathcal{K}_K$ defines a fibration $\mathrm{Diff}^+(S^3) \to \mathcal{K}_K$ with fiber $\mathrm{Diff}^+(S^3, K)$ the diffeomorphisms f with $f(K) = K$. The action of $\mathrm{Diff}^+(S^3)$ restricts to an action of the subgroup $SO(4) \subset \mathrm{Diff}^+(S^3)$, and the orbit of K is the coset space $SO(4)/G_K$ where G_K is the symmetry group G_K of K. Thus we have a commutative diagram of fibrations

$$\begin{tikzcd}
\mathrm{Diff}^+(S^3,K) \arrow[r] & \mathrm{Diff}^+ S^3 \arrow[r] & \mathcal{K}_K \\
G_K \arrow[r] \arrow[u, hook] & SO(4) \arrow[r] \arrow[u, hook] & SO(4)/G_K \arrow[u, hook]
\end{tikzcd}$$

We will show that if K is a torus knot or a hyperbolic knot satisfying the linearization conjecture then K can be positioned in S^3 so that the inclusion $G_K \hookrightarrow \mathrm{Diff}^+(S^3, K)$ induces isomorphisms on all homotopy groups. By the Smale conjecture the inclusion $SO(4) \hookrightarrow \mathrm{Diff}^+(S^3)$ also induces isomorphisms of homotopy groups. The five-lemma will then imply that the inclusion of the model $\mathcal{M}_K = SO(4)/G_K$ into $\mathcal{K}_K$ induces isomorphisms on homotopy groups, so is a homotopy equivalence.

To analyze $\mathrm{Diff}^+(S^3, K)$ we will use the spaces in the following diagram:

$$\begin{tikzcd}
\mathrm{Diff}^+(S^3,K) & p^{-1}(\mathrm{I}^+(\partial M)) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "p"] & \mathrm{Diff}^+_E(M) \arrow[r, hook] \arrow[d, "p"] & \mathrm{Diff}^+(M) \\
 & \mathrm{I}^+(\partial M) \arrow[r, hook] & \mathrm{Diff}^+_{\pm I}(\partial M) &
\end{tikzcd}$$

Here $M \subset S^3$ is the complement of the interior of a solid torus tubular neighborhood $N(K)$ of K. The orientation-preserving diffeomorphisms of M that extend to diffeomorphisms of S^3 form a subgroup $\mathrm{Diff}^+_E(M) \subset \mathrm{Diff}^+(M)$ that is a union of path-components of $\mathrm{Diff}^+(M)$, the diffeomorphisms taking meridian circles in ∂M to meridians, up to sign and isotopy in ∂M. By [GL] this subgroup is in fact the whole group, but we will not make essential use of this fact. Elements of $\mathrm{Diff}^+(M)$ must take longitudes to longitudes, up to sign and isotopy, for homological reasons. Restricting elements of $\mathrm{Diff}^+_E(M)$ to ∂M, we thus obtain a fibration $p : \mathrm{Diff}^+_E(M) \to \mathrm{Diff}^+_{\pm I}(\partial M)$ whose base space is the group of orientation-preserving diffeomorphisms of the torus ∂M whose action on π_1 is plus or minus the identity. This base space deformation retracts onto the subspace $I^+(\partial M)$ consisting of orientation-preserving diffeomorphisms that rotate or reflect each factor of $\partial M = S^1 \times S^1$ separately. Thus each of the two components of $I^+(\partial M)$ is homeomorphic to a torus. The deformation retraction of $\mathrm{Diff}^+_{\pm I}(\partial M)$ onto $I^+(\partial M)$ lifts to a homotopy equivalence of $\mathrm{Diff}^+_E(M)$ with $p^{-1}(I^+(\partial M))$. Elements of $p^{-1}(I^+(\partial M))$ extend canonically over $N(K)$ by rotations or reflections of the factors of $N(K)$, giving an inclusion $p^{-1}(I^+(\partial M)) \hookrightarrow \mathrm{Diff}^+(S^3, K)$, and this inclusion is a homotopy equivalence. Namely, by standard differential topology techniques, $\mathrm{Diff}^+(S^3, K)$ deformation retracts onto the subspace of diffeomorphisms taking $N(K)$ to itself by diffeomorphisms that take fiber disks of $N(K)$ to fiber disks by rotations or reflections. Then by further deformation retractions we can arrange that K goes to itself by rotation or reflection and that the diffeomorphism of $N(K)$ is a product of rotations or reflections in its two factors.

In case K is a nontrivial torus knot in its standard position the symmetry group G_K is a copy of $O(2)$ which we can view as a subgroup of $\mathrm{Diff}^+(M)$, and it follows from [H2], [I] that $\mathrm{Diff}^+(M)$ deformation retracts onto this subgroup. This can be seen from the Seifert fibering of M, whis has base a disk and two multiple fibers, of distinct multiplicities. There is an annulus in M separating it into two solid tori, and $\mathrm{Diff}^+(M)$ deformation retracts onto the subspace leaving this annulus invariant, and a further deformation retraction takes one to the subspace of diffeomorphisms taking fibers to fibers, from which the result easily follows.

Thus when K is a nontrivial torus knot in standard position we have the inclusion $G_K \hookrightarrow \mathrm{Diff}^+(S^3, K)$ a homotopy equivalence.

Assume now that K is a hyperbolic knot. In this case Mostow rigidity implies that the composition $\mathrm{Isom}(S^3 - K) \hookrightarrow \mathrm{Diff}(M) \hookrightarrow \mathrm{Out}(\pi_1 M)$ of the isometry group of the hyperbolic manifold $S^3 - K$ into the outer automorphism group of $\pi_1 M$ is an isomorphism. The second inclusion is a homotopy equivalence, by [H2], [I], using the fact that the space of homotopy equivalences $M \to M$ deformation retracts onto the space of homotopy equivalences of pairs $(M, \partial M) \to (M, \partial M)$, which follows from the fact that every $\mathbb{Z}\times\mathbb{Z}$ subgroup of $\pi_1 M$ is contained in $\pi_1(\partial M)$, up to conjugation, since K is hyperbolic. The inclusion $\mathrm{Isom}(S^3 - K) \hookrightarrow \mathrm{Diff}(M)$ is then a homotopy equivalence, and this remains true also for the subgroups $\Gamma_K = \mathrm{Isom}^+(S^3 - K)$ and $\mathrm{Diff}^+(M)$ of index at most two consisting of orientation-preserving isometries or diffeomorphisms.

If K satisfies the linearization conjecture, as described in the introduction, then the subgroup $\Gamma_K \hookrightarrow \mathrm{Diff}^+(S^3, K)$ is conjugate via an element $\sigma \in \mathrm{Diff}^+(S^3)$ to a subgroup of $SO(4)$. Replacing K by the equivalent knot $\sigma(K)$, we may assume that Γ_K itself is a subgroup of $SO(4)$. Thus Γ_K is a subgroup of the symmetry group G_K, and in fact it must be equal to G_K because the natural map $G_K \to \pi_0 \mathrm{Diff}^+(S^3 - K) \approx \Gamma_K$ is injective by the theorem that a periodic diffeomorphism of a Haken manifold which is homotopic to the identity must be part of an S^1 action (see [T],[FY]), and this happens only when $S^3 - K$ is a Seifert manifold, hence K is a torus knot.

Spaces of Arcs

Knotted arcs in $\mathcal{A}$ can be given a preferred orientation since the endpoints of these arcs are fixed. This leads us to consider the two-sheeted covering space $\mathcal{K}^+$ of $\mathcal{K}$ consisting of knots with a choice of orientation. Path-components of $\mathcal{A}$ and $\mathcal{K}^+$ are in one-to-one correspondence. Let $\mathcal{A}_K$ and $\mathcal{K}_K^+$ be the path-components corresponding to a given oriented knot K. If K is an invertible knot then $\mathcal{K}_K^+$ is a two-sheeted cover of the corresponding component $\mathcal{K}_K$ of $\mathcal{K}$, and otherwise $\mathcal{K}_K^+$ is homeomorphic to $\mathcal{K}_K$.

The space of arcs $\mathcal{A}_K$ has the same homotopy type as the subspace $\mathcal{K}_K^0$ of $\mathcal{K}_K^+$ consisting of knots passing through a given point of S^3 and having a given unit tangent vector at that point, pointing in the direction of the orientation of the knot. Let us take the given point and tangent direction to be the first two standard basis vectors e_1 and e_2 in $\mathbb{R}^4$. Let $\mathcal{K}_K^{fr}$ be the space of pairs consisting of an oriented knot K' isotopic to K and a positively-oriented orthonormal tangent frame to S^3 at a point of K', whose first vector is tangent to K' pointing in the direction of the orientation. There is a homeomorphism $SO(4)\times\mathcal{K}_K^0 \approx \mathcal{K}_K^{fr}$ sending a pair (ρ, K') in the product to the knot $\rho(K')$ with the framing at $\rho(e_1)$ consisting of ρ of the standard basis vectors e_2, e_3, e_4.

There is a projection $\mathcal{K}_K^{fr} \to \mathcal{K}_K^+$ by forgetting the frame except for the orientation of K induced by its first vector. This projection is a fibration, and the fibers are tori. Denote by $\mathcal{M}_K^+$ the preimage in $\mathcal{K}_K^+$ of an $SO(4)$-invariant subspace $\mathcal{M}_K \subset \mathcal{K}_K$. Then we have fibrations

$$\begin{array}{ccccc} T & \longrightarrow & \mathcal{K}_K^{fr} & \longrightarrow & \mathcal{K}_K^{+} \\ \| & & \uparrow & & \uparrow \\ T & \longrightarrow & \mathcal{M}_K^{fr} & \longrightarrow & \mathcal{M}_K^{+} \end{array} \qquad \begin{array}{ccccc} SO(4) & \longrightarrow & \mathcal{K}_K^{fr} & \longrightarrow & \mathcal{K}_K^{0} \\ \| & & \uparrow & & \uparrow \\ SO(4) & \longrightarrow & \mathcal{M}_K^{fr} & \longrightarrow & \mathcal{M}_K^{0} \end{array}$$

where $\mathcal{M}_K^{fr}$ is the preimage of $\mathcal{M}_K^+$ in $\mathcal{K}_K^{fr}$ and $\mathcal{M}_K^0$ is the image of $\mathcal{M}_K^{fr}$ in $\mathcal{K}_K^0$. If the inclusion $\mathcal{M}_K^+ \hookrightarrow \mathcal{K}_K^+$ is a weak homotopy equivalence, then so will be the other inclusion in the left-hand diagram, and therefore also the inclusion $\mathcal{M}_K^0 \hookrightarrow \mathcal{K}_K^0$.

In particular, if K is a torus knot or a hyperbolic knot satisfying the linearization conjecture, we obtain the results about a model for $\mathcal{A}_K$ stated in the introduction.

2. Satellite Knots

A knot K is a satellite knot if it lies in a solid torus tubular neighborhood V of another nontrivial knot in such a way that it cannot be isotoped within V to be disjoint from a meridian disk of V. This is equivalent to saying that the torus ∂V is incompressible in $S^3 - K$. By a classical theorem of Alexander, every torus in S^3 bounds a solid torus on one side or the other, so studying the various ways in which K can be a satellite is equivalent to studying incompressible tori in $S^3 - K$. The leads one to consider the canonical torus decomposition of $S^3 - K$. This is a minimal collection T of finitely many disjoint incompressible tori T_i in $S^3 - K$ splitting $S^3 - K$ into pieces that are either Seifert fibered or atoroidal — containing only boundary-parallel incompressible tori. The minimality of T with respect to inclusion guarantees that it is unique up to isotopy. See [H] or [NS] for elementary expositions.

It is known that Seifert complementary pieces of the torus decomposition can only be of rather limited types. A trivial possibility that we are not interested in here is that K is a torus knot, with T empty and $S^3 - K$ itself Seifert fibered. Apart from this, there are just two other possibilities for Seifert pieces in the torus decomposition of a knot:

(1) A cable piece. This is obtained from a solid torus with one of its standard Seifert fiberings with one multple fiber by deleting a regular fiber, or a fibered tubular neigborhood of this regular fiber. If the solid torus is knotted in S^3, then the regular fiber is a satellite of the core knot of the solid torus, namely a cable on this knot. More generally one could put a more complicated knot in the fibered neighborhood of the regular fiber, and this knot would also be a satellite of the core knot of the solid torus.

(2) A sum piece. This is a product of S^1 and a disk with at least two punctures, with the product fibering. In particular there are no multiple fibers, and there are at least three boundary tori. This configuration arises when one forms connected sums of knots. Since it is somewhat more complicated to describe than the case of cable pieces, we postpone a detailed discussion until later.

Associated to a torus decomposition T for $S^3 - K$ is a graph τ_K having a vertex for each complementary piece and an edge for each torus in T. Since every closed surface in S^3 separates, τ_K is a tree. It has a distinguished vertex corresponding to the piece abutting K. Orienting the edges in the direction of this vertex is the same as giving the tori of T normal orientations toward the solid tori they bound.

The Special Case of an Unbranched Tree

The easiest case is that τ_K is homeomorphic to a line segment, with no branching. This says that the torus decomposition T consists of tori $T_1, \cdots, T_n$ bounding solid tori $V_1 \supset \cdots \supset V_n$. For each V_i we choose a product structure $V_i \approx S^1 \times D^2$. Let X_K be the product of n tori $S^1 \times S^1$. We view points $\theta \in X_K$ as n-tuples of rotations θ_i of V_i, the two S^1 coordinates of θ_i specifying the angles of longitudinal and meridional rotation of V_i. For each $\theta \in X_K$ we construct a function $f_\theta : S^3 \to S^3$ by first rotating V_n by θ_n, then V_{n-1} by θ_{n-1}, and so on, ending with the identity outside V_1. Thus on $V_i - V_{i+1}$, f_θ is the composition $\theta_1 \cdots \theta_i$. The function f_θ is discontinuous along T_i, but this is not a problem if we restrict to K, so we obtain a map $X_K \to \mathcal{K}_K$, $\theta \mapsto f_\theta(K)$. More generally, we will be interested in the map $SO(4) \times X_K \to \mathcal{K}_K$, $(\rho, \theta) \mapsto \rho(f_\theta(K))$.

Now let us bring symmetries into the picture. Symmetries of the torus and the solid torus will play an important role. In particular there are the orientation-preserving symmetries of $S^1 \times D^2$ or $S^1 \times S^1$ of the form $(x, y) \mapsto (r_1(x), r_2(y))$ where r_1 and r_2 are rotations or reflections of the two factors. These form a group I^+ which we we think of as either isometries or linear symmetries. The group I^+ has a subgroup I_0^+ of index 2 consisting of pairs (r_1, r_2) of rotations. The remaining elements of I^+ reverse orientations of longitudes and meridians, fixing two line segments in the solid torus, or four points on the boundary torus. If we embed the solid torus in S^3 in a standard way, these symmetries are realizable by rotations of S^3.

Returning to the satellite knot K, if the knot K_1 formed by the core circle of V_1 satisfies the linearization conjecture, we may isotope it into symmetric position, with symmetry group $\Gamma_0 \subset SO(4)$ isomorphic to $\pi_0\mathrm{Diff}^+(S^3, K_1)$ if $S^3 - K_1$ is hyperbolic, or to $O(2)$ in the case that K_1 is a torus knot with $S^3 - K_1$ Seifert fibered. We may choose the tubular neighborhood V_1 of K_1 to be a small ε-neighborhood invariant under Γ_0, and we may choose the product structure $V_1 \approx S^1 \times D^2$ so that Γ_0 acts on this product via elements of I^+. Next we have the symmetry group Γ_1 of (V_1, K_2) for K_2 the core circle of V_2, so Γ_1 is $\pi_0\mathrm{Diff}^+(V_1, K_2)$ if $V_1 - K_2$ is hyperbolic or $O(2)$ if $V_1 - K_2$ is Seifert fibered, a cable space. It is a classical fact that an orientation-preserving action of a finite group on the solid torus V_1 is equivalent to an action by a subgroup of I^+. So after an isotopy of K_2 in V_1 we may assume Γ_1 acts on (V_1, K_2) via elements of I^+. As before we may choose V_2 nicely so that Γ_1 acts on it as well by elements of I^+. In the same fashion we continue on with the symmetry groups $\Gamma_2, \cdots, \Gamma_n$, at the last stage isotoping K in V_n to linearize the action of Γ_n.

Elements of Γ_i are orientation-preserving diffeomorphisms that take meridians to meridians and longitudes to longitudes in T_i and T_{i+1}, possibly reversing orientations of these meridians and longitudes. Let Γ_K be the subgroup of $\Gamma_0 \times \cdots \times \Gamma_n$ consisting of tuples $(g_0, \cdots, g_n)$ such that the restrictions of g_i and g_{i-1} to T_i are isotopic for each i. This is equivalent to requiring g_i and g_{i-1} both to preserve or both to reverse orientations of meridians and longitudes in T_i. We can define an action of Γ_K on $SO(4) \times X_K$ by sending $(\rho, \theta_1, \cdots, \theta_n)$ to $(\rho g_0^{-1}, g_0\theta_1 g_1^{-1}, \cdots, g_{n-1}\theta_n g_n^{-1})$. This is evidently a free action. The map $SO(4) \times X_K \to \mathcal{K}_K$, $(\rho, \theta) \mapsto \rho(f_\theta(K))$, is constant on orbits since $(\rho g_0^{-1})(g_0\theta_1 g_1^{-1}) \cdots (g_{n-1}\theta_n g_n^{-1})(K) = \rho\theta_1 \cdots \theta_n(K)$, using the fact that $g_n(K) = K$ and regarding each g_i for $i < n$ as a diffeomorphism of (V_i, V_{i+1}), or of (S^3, V_1) for $i = 0$.

Thus we obtain a map $(SO(4) \times X_K)/\Gamma_K \to \mathcal{K}_K$.

Theorem 1. *This map $(SO(4) \times X_K)/\Gamma_K \to \mathcal{K}_K$ is a homotopy equivalence.*

Proof: First let us do something about the discontinuities of f_θ along the T_i's. If we split S^3 along T we obtain compact manifolds $W_0, \cdots, W_n$ with W_0 the closure of $S^3 - V_1$, W_i the closure of $V_i - V_{i+1}$ for $0 < i < n$, and $W_n = V_n$. The function f_θ determines a continuous map from the disjoint union of the W_i's to S^3. If we identify points in this disjoint union having the same image in S^3 then we obtain a 3-manifold S^3_θ and an induced diffeomorphism $S^3_\theta \to S^3$. Explicitly, we are identifying each point $x \in T_i \subset W_i$ with $\theta_i(x) \in T_i \subset W_{i-1}$.

Let $\mathrm{DiffExt}^+(S^3)$ be the space of pairs (θ, f) where $\theta \in X_K$ and $f : S^3_\theta \to S^3$ is an orientation-preserving diffeomorphism. We think of f as an 'exterior diffeomorphism.' By writing f as the composition of the canonical f_θ defined earlier and a diffeomorphism $S^3 \to S^3$ we see that $\mathrm{DiffExt}^+(S^3)$ is homeomorphic to the product of X_K and $\mathrm{Diff}^+(S^3)$. We can in fact use this bijection to define the topology on

$\mathrm{DiffExt}^+(S^3)$ if we like. We will fit the space $\mathrm{DiffExt}^+(S^3)$ into a commutative diagram

$$\begin{tikzcd}
\mathrm{Diff\,Ext}^+(S^3, K) \arrow[r] & \mathrm{Diff\,Ext}^+ S^3 \arrow[r] & \mathcal{K}_K \\
\Gamma_K \arrow[r] \arrow[u, hook] & SO(4) \times X_K \arrow[r] \arrow[u, hook] & (SO(4) \times X_K)/\Gamma_K \arrow[u, hook]
\end{tikzcd}$$

The map $\mathrm{DiffExt}^+(S^3) \to \mathcal{K}_K$ sends (θ, f) to $f(K)$. This is a fibration whose fiber $\mathrm{DiffExt}^+(S^3, K)$ is the subspace consisting of pairs (θ, f) with $f(K) = K$. The middle vertical inclusion in the diagram sends (ρ, θ) to $(\theta, \rho f_\theta)$, so this is equivalent to the natural inclusion $SO(4) \times X_K \hookrightarrow \mathrm{Diff}^+(S^3) \times X_K$, which is a homotopy equivalence by the Smale conjecture.

Next we analyze the fiber $\mathrm{DiffExt}^+(S^3, K)$. This has a natural projection to the space of torus decompositions of $S^3 - K$, the map $(\theta, f) \mapsto f(T)$, where we regard the given torus decomposition T as a decomposition of $S^3_\theta - K$. The space of torus decompositions is contractible by [H2] (this is made more explicit in the revised version), so the inclusion of the fiber $\mathrm{DiffExt}^+(S^3, K \cup T)$ into $\mathrm{DiffExt}^+(S^3, K)$ is a homotopy equivalence. We can regard $\mathrm{DiffExt}^+(S^3, K \cup T)$ as the space of tuples $(f_0, \cdots, f_n)$ of orientation-preserving diffeomorphisms $f_i : W_i \to W_i$ whose restrictions to common boundary tori agree up to rotation, an element of I_0^+. (The f_i also take meridians to meridians and longitudes to longitudes, up to isotopy and sign.) From this viewpoint there is an evident inclusion $\Gamma_K \hookrightarrow \mathrm{DiffExt}^+(S^3, K \cup T)$, and this is a homotopy equivalence. Namely we can first deform the f_i's into Γ_i by [H2] and then adjust these deformations so that they agree modulo I_0^+ on the T_i's.

It is evident that the diagram commutes, so since the first two vertical maps are homotopy equivalences, so is the third. □

Satellite Knots Without Sums

[Not yet written]

General Satellite Knots

[Not yet written]

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