Math 2940: Determinants and row operations

Theorem 3 in Section 3.2 describes how the determinant of a matrix changes when row operations are performed. The proof given in the textbook is somewhat obscure, so this handout provides an alternative proof.

Theorem. *Let A be a square matrix.*

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

In order to prove the theorem, it is convenient to define the *cofactors* of a matrix. If the $n \times n$ matrix A has entries a_{ij}, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

then the (i, j) cofactor C_{ij} is equal to the determinant of the $(n-1) \times (n-1)$ submatrix of A with row i and column j removed, multiplied by the sign $(-1)^{i+j}$. As an example, if

$$A = \begin{bmatrix} -1 & 3 & -2 & 0 \\ 4 & -2 & 0 & 3 \\ -2 & -2 & 1 & 4 \\ -1 & -4 & 0 & 1 \end{bmatrix},$$

then the $(2, 3)$ cofactor of A is

$$C_{23} = -\begin{vmatrix} -1 & 3 & 0 \\ -2 & -2 & 4 \\ -1 & -4 & 1 \end{vmatrix}.$$

Suppose we choose to compute the determinant of A by expanding along the ith row $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$. Then

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

Proof of part c. Suppose

$$A = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \qquad B = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ k \cdot a_{i1} & k \cdot a_{i2} & \cdots & k \cdot a_{in} \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

The cofactors in the ith row are the same for both matrices, so

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}, \qquad \det B = k \cdot a_{i1}C_{i1} + \cdots + k \cdot a_{in}C_{in}.$$

This shows that $\det B = k \cdot \det A$. □

Proof of part b. First suppose that the two interchanged rows are consecutive:

$$A = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \qquad B = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ w_1 & w_2 & \cdots & w_n \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

If both $\det A$ and $\det B$ are computed by expanding along the $\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$ row, then all the cofactors are the same except that the signs are flipped. Therefore, $\det B = -\det A$.

Now suppose that the two interchanged rows $i < i'$ are not consecutive.

$$A = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \qquad B = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

We can transform A into B by performing an *odd* number of consecutive row interchanges, each of which multiplies the determinant by -1. Therefore,

$$\det B = (-1)^{(\text{odd } \#)} \det A = -\det A.$$

First we move $\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ down from row i to row i' one step at a time. This takes $i' - i$ interchanges of consecutive rows. Afterwards, $\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ is in the right place, while $\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$ is in row $i' - 1$. Therefore we must move $\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$ up from row $i' - 1$ to row i, which takes $i' - i - 1$ interchanges of consecutive rows. The total number of interchanges is

$$(i' - i) + (i' - i - 1) = 2(i' - i) - 1,$$

which is an odd number. □

Suppose that A contains two copies of the same row, that is, $\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$ in the matrices above. Then the matrix B, in which the rows have been interchanged, is the same as A. The argument given above shows that $\det A = -\det A$, meaning that $\det A = 0$. Therefore any matrix that contains two copies of the same row must have determinant zero.

Proof of part a. Let

$$A = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \qquad B = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ w_1 + k \cdot v_1 & w_2 + k \cdot v_2 & \cdots & w_n + k \cdot v_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

Suppose that $\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$ (in A) and $\begin{bmatrix} w_1 + k \cdot v_1 & \cdots & w_n + k \cdot v_n \end{bmatrix}$ (in B) are in row i. Then

$$\det A = w_1 C_{i1} + \cdots + w_n C_{in}$$

and

$$\begin{aligned} \det B &= (w_1 + k \cdot v_1) C_{i1} + \cdots + (w_n + k \cdot v_n) C_{in} \\ &= (w_1 C_{i1} + \cdots + w_n C_{in}) + k \cdot (v_1 C_{i1} + \cdots + v_n C_{in}). \end{aligned}$$

This means that

$$\det B = \det A + k \cdot \det \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

The last matrix has two copies of the same row, so its determinant is zero by the remark at the bottom of the previous page. We conclude that $\det B = \det A + k \cdot 0 = \det A$. $\square$