Math 2940: Final Exam Practice Solutions

1. (a) Write in parametric vector form the solutions to the equation

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 2 & 5 & -1 \\ 2 & 4 & 1 & 1 \\ 4 & 8 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 21 \end{bmatrix}.$$

Solution: Row-reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 6 \\ 1 & 2 & 5 & -1 & 0 \\ 2 & 4 & 1 & 1 & 9 \\ 4 & 8 & -1 & 2 & 21 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 6 \\ 0 & 0 & 6 & -2 & -6 \\ 0 & 0 & 3 & -1 & -3 \\ 0 & 0 & 3 & -2 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 6 \\ 0 & 0 & 3 & -1 & -3 \\ 0 & 0 & 3 & -1 & -3 \\ 0 & 0 & 3 & -2 & -3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 6 \\ 0 & 0 & 3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 6 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 6 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The equations are:

$$\begin{aligned} x_1 &= 5 - 2x_2 \\ x_2 &= x_2 \\ x_3 &= -1 \\ x_4 &= 0 \end{aligned}$$

so the solutions have the form

$$\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Let A be the 4×4 matrix from part (a). Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$.

Solution: From part (a) we have

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for Nul(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. A basis for Col($A$) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

2. Let

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 4 & -4 \\ 0 & 1 & 3 \end{bmatrix}.$$

A student has computed that

$$A^{-1} = \begin{bmatrix} 4 & 8 & -1 \\ -1 & -2 & 3 \\ -10 & -20 & 7 \end{bmatrix}.$$

Without actually computing A^{-1} yourself, explain why the student is wrong.

Solution: We multiply A by the purported matrix A^{-1} and see whether we get the identity matrix. Just looking at the top left entry,

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 4 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 8 & -1 \\ -1 & -2 & 3 \\ -10 & -20 & 7 \end{bmatrix} = \begin{bmatrix} 33 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

so the product cannot be the identity matrix.

3. For each of the following properties, determine whether an $n \times n$ matrix A satisfying the property is always invertible, sometimes invertible, or never invertible. Briefly explain your reasoning.

(a) There is at least one solution to $A\mathbf{x} = \mathbf{0}$.

Answer: Sometimes invertible. It's always true that there is at least one solution to $A\mathbf{x} = \mathbf{0}$, namely $\mathbf{x} = \mathbf{0}$, so all $n \times n$ matrices A satisfy this property.

(b) For every $\mathbf{b}$ in $\mathbf{R}^n$, there is at least one solution to $A\mathbf{x} = \mathbf{b}$.

Answer: Always invertible. The property says that Col(A) $= \mathbf{R}^n$, so A has rank n and is therefore invertible.

(c) There is at most one solution to $A\mathbf{x} = \mathbf{0}$.

Answer: Always invertible. The property says that Nul(A) $= \{\mathbf{0}\}$, which also implies that A has rank n.

(d) There is a vector $\mathbf{b}$ in $\mathbf{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Answer: Always invertible. Given that there is at least one vector $\mathbf{x}_0$ such that $A\mathbf{x}_0 = \mathbf{b}$, for any $\mathbf{y} \in \text{Nul}(A)$ we also have $A(\mathbf{x}_0 + \mathbf{y}) = \mathbf{b}$. The property that there is only one solution to $A\mathbf{x} = \mathbf{b}$ means that $\text{Nul}(A)$ contains no nonzero vectors, so we are in the situation of part (c).

4. Let A be an $n \times n$ matrix. Find a formula for $\det(-A)$ in terms of $\det(A)$.

Solution: We can get from A to $-A$ by multiplying each row by -1. Every time we do this, the determinant gets multiplied by -1. Therefore, $\det(-A) = (-1)^n \det(A)$.

5. Let A be a 5×3 matrix. Suppose that

$$A \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = A \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 4 \end{bmatrix}.$$

(a) What are the possible values of $\dim \text{Nul}(A)$ and $\dim \text{Col}(A)$?

Solution: The null space is a subspace of $\mathbf{R}^3$, so the initial possibilities for its dimension are $0, 1, 2, 3$. We know that

$$A\left(\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so $\begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \in \text{Nul}(A)$. This rules out dimension 0. Dimension 3 is also impossible because that would mean that $\text{Nul}(A) = \mathbf{R}^3$, which is only possible when A is the zero matrix. Both dimensions 1 and 2 are possible.

Since $\dim \text{Col}(A) = 3 - \dim \text{Nul}(A)$, it could be 2 or 1.

(b) If the first column of A is $\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, show that $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \notin \text{Col}(A)$.

Solution: The column space includes the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

From part (a) it has dimension at most 2, so the dimension equals 2 and $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$. The vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ since its fourth coordinate is nonzero, so it isn't in $\mathrm{Col}(A)$.

6. Let $\mathcal{M}_{2\times 2}$ be the space of 2×2 matrices. It is true (you do not have to prove) that $\mathcal{M}_{2\times 2}$ is a vector space: we know how to add 2×2 matrices to each other and multiply by scalars, and there is a zero matrix. Define a function $S : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $S(A) = A + A^T$.

(a) Verify that the four matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are a basis for $\mathcal{M}_{2\times 2}$.

Solution: We check that these matrices are linearly independent and that they span $\mathcal{M}_{2\times 2}$. Linear independence: If

$$c_1E_1 + c_2E_2 + c_3E_3 + c_4E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $c_1 = c_2 = c_3 = c_4 = 0$.

Spanning: Any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}$ can be written as the linear combination

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_1 + bE_2 + cE_3 + dE_4.$$

(b) Show that S is a linear transformation.

Solution: We must check that $S(A + B) = S(A) + S(B)$ and $S(cA) = cS(A)$. For the first property,

$$S(A + B) = (A + B) + (A + B)^T = (A + A^T) + (B + B^T) = S(A) + S(B).$$

For the second property,

$$S(cA) = cA = (cA)^T = c(A + A^T) = cS(A).$$

(c) Find a basis for the kernel of S.

Solution: The kernel of S is the set of matrices A such that $A + A^T$ is the zero matrix. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, that means

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so the requirements are that $a = 0$, $b + c = 0$, and $d = 0$. We can view c as a free variable, and then

$$\begin{aligned} a &= 0 \\ b &= -c \\ c &= c \\ d &= 0 \end{aligned}$$

which means that every matrix in the kernel of S can be written as $c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Hence $\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for the kernel of S.

(d) Find a basis for the range of S.

Solution: The range of S is all the matrices of the form $\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$. In fact this is simply the space of all symmetric 2×2 matrices: clearly any matrix in the range of S must be symmetric, and any symmetric matrix can be written in this form by letting a be one-half the top left entry, d be one-half the bottom right entry, and b, c be any numbers that add up to the off-diagonal entry. A basis is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. Clearly these matrices are contained in the range of S. They are linearly independent by the same type of reasoning used to prove linear independence in (a), and they span the range of S because of the decomposition

$$\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = 2a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b+c)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

7. Prove that the determinant of an $n \times n$ matrix A is the product of the eigenvalues (counted according to their algebraic multiplicities). *Hint:* Write

the characteristic polynomial as

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Solution: If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ (counted with algebraic multiplicity), then as the hint says, the characteristic polynomial of A is

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Plugging in $\lambda = 0$ yields $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$.

8. Consider the Markov transition matrix

$$A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}.$$

(a) Because A is a Markov transition matrix, it has a steady-state vector. This immediately implies that A has a certain eigenvalue. What is that eigenvalue?

Solution: The steady-state vector $\mathbf{v}$ satisfies $A\mathbf{v} = \mathbf{v}$, thus it is an eigenvector with eigenvalue 1.

(b) Use part (a) along with Problem 7 to find the other eigenvalue of A. Then find a diagonalization $A = PDP^{-1}$.

Solution: The product of the eigenvalues is $\det A = 0.48 - 0.08 = 0.4$. Since one of the eigenvalues is 1, the other must be 0.4.

The $\lambda = 1$ eigenspace is $\text{Nul}(A - I)$:

$$\begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

so the eigenspace is spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Incidentally, this means that the steady-state vector is $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

The $\lambda = 0.4$ eigenspace is $\text{Nul}(A - 0.4I)$:

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so the eigenspace is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

A diagonalization for A is $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}.$$

(c) Consider the discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$. If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find a formula for $\mathbf{x}_k$.

Solution: The general formula is $\mathbf{x}_k = c_1 1^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2(0.4)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. When $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

This means $\mathbf{x}_k = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3}(0.4)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(d) Provide an interpretation of your answer to part (c) in terms of probabilities of the Markov chain.

Solution: The initial vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ means that at time 0, the Markov chain is in state 1 with probability 1 and state 2 with probability 0. The vector $\mathbf{x}_k = A^k\mathbf{x}_0$ provides the probabilities of being in states 1 and 2 at time k. The formula from part (c)

$$\mathbf{x}_k = \begin{bmatrix} (1/3) + (2/3)(0.4)^k \\ (2/3) - (2/3)(0.4)^k \end{bmatrix}$$

means that at time k, the probability of being in state 1 is $\frac{1}{3} + \frac{2}{3}(0.4)^k$, while the probability of being in state 2 is $\frac{2}{3} - \frac{2}{3}(0.4)^k$. Note that at time $k = 0$ the probabilities are 1 and 0 respectively, as they should be. For all positive values of k, the probabilities are positive numbers that sum to 1. In the long run as $k \to \infty$, the probabilities tend to the steady-state probabilities 1/3 and 2/3.

9. Construct three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbf{R}^2$ such that the dot products $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$ are all negative.

Solution: The idea here is to find three vectors such that the angle between each pair is greater than 90 degrees. For example, we could let

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Graphing $\mathbf{a}, \mathbf{b}, \mathbf{c}$ verifies that the angles are all greater than 90 degrees, and indeed the dot products are all negative. Note: it would not be possible to do this with 4 or more vectors in $\mathbf{R}^2$!

10. Let A be an $m \times n$ matrix whose columns are orthonormal vectors. Prove that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

Solution: It is easier (and equivalent) to show that $\|A\mathbf{x}\|^2 = \|\mathbf{x}\|^2$. The assumption that A has orthonormal columns is equivalent to saying that $A^T A = I$. With this in mind,

$$\|A\mathbf{x}\|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = (A\mathbf{x})^T (A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2.$$

11. Find the QR factorization of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be the columns of A in order. Run Gram-Schmidt to get an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for Col(A). First, $\mathbf{v}_1 = \mathbf{a}_1$. Second,

$$\mathbf{a}_2 - \left(\frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$$

so we can let $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$. For future reference, $\mathbf{a}_2 - \frac{1}{2}\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2$. Third,

$$\mathbf{a}_3 - \left(\frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$

so we let $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$ and note that $\mathbf{a}_3 - \frac{1}{2}\mathbf{v}_1 - \frac{1}{6}\mathbf{v}_2 = \frac{1}{3}\mathbf{v}_3$.

The lengths of the $\mathbf{v}_i$ are $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{v}_2\| = \sqrt{6}$, $\|\mathbf{v}_3\| = \sqrt{12}$. Let $\mathbf{u}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$ be the normalized vectors, so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for Col(A). Meanwhile, since $\mathbf{v}_i = \|\mathbf{v}_i\|\mathbf{u}_i$, we have

$$\begin{aligned} \mathbf{a}_1 &= \sqrt{2}\mathbf{u}_1, \\ \mathbf{a}_2 &= \frac{\sqrt{2}}{2}\mathbf{u}_1 + \frac{\sqrt{6}}{2}\mathbf{u}_2, \\ \mathbf{a}_3 &= \frac{\sqrt{2}}{2}\mathbf{u}_1 + \frac{\sqrt{6}}{6}\mathbf{u}_2 + \frac{\sqrt{12}}{3}\mathbf{u}_3. \end{aligned}$$

Hence the QR factorization is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 & \sqrt{6}/6 \\ 0 & 0 & \sqrt{12}/3 \end{bmatrix}.$$

12. Let $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$. (Note that the rows of A are orthogonal.) If $\mathbf{x} = \begin{bmatrix} -6 \\ 2 \\ -1 \end{bmatrix}$, find $\mathbf{u}$ in Row(A) and $\mathbf{v}$ in Nul(A) such that $\mathbf{x} = \mathbf{u} + \mathbf{v}$.

Solution: Let $\mathbf{a}_1, \mathbf{a}_2$ be the rows of A. The orthogonal projection of $\mathbf{x}$ onto Row(A) is

$$\mathbf{u} = \left(\frac{\mathbf{x} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}\right) \mathbf{a}_1 + \left(\frac{\mathbf{x} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2}\right) \mathbf{a}_2 = \frac{5}{5} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \frac{-6}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.$$

The difference $\mathbf{v} = \mathbf{x} - \mathbf{u} = \begin{bmatrix} -6 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix}$ is in Nul(A), since Nul(A) is the orthogonal complement of Row(A).

13. Donald and Hillary are arguing about planes through the origin in $\mathbf{R}^3$. Donald claims that a plane through the origin is the span of two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Hillary claims that a plane through the origin is the set of all points

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

that satisfy an equation $ax_1 + bx_2 + cx_3 = 0$, where the coefficients a, b, c are not all zero. Of course both of them are correct.

(a) Explain why Hillary's definition describes the subspace of $\mathbf{R}^3$ that is orthogonal to Span $\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$. In the case $a = 1$, $b = -3$, $c = 2$, find vectors $\mathbf{v}_1, \mathbf{v}_2$ that satisfy Donald's definition.

Solution: The equation $ax_1 + bx_2 + cx_3 = 0$ is equivalent to $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. Thus the equation is satisfied for vectors $\mathbf{x}$ that are orthogonal to the span of

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

It's also true that the space of solutions to $ax_1+bx_2+cx_3 = 0$ is the null space of the matrix $\begin{bmatrix} a & b & c \end{bmatrix}$. For the given values of a, b, c, we have $\begin{bmatrix} 1 & 3 & -2 \end{bmatrix}$, which is already in reduced row echelon form and gives the equation $x_1-3x_2+2x_3 = 0$. It may seem like this is going in circles, but now we can use the established procedure to find a basis of the null space:

$$\begin{aligned} x_1 &= 3x_2 - 2x_3 \\ x_2 &= x_2 \\ x_3 &= \qquad\quad x_3 \end{aligned}$$

so $\mathbf{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Therefore we can take $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

(b) Now suppose that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}.$$

Find coefficients a, b, c that satisfy Hillary's definition.

Solution 1: The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ should be orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, by the discussion in part (a). That means it lies in the null space of the matrix whose rows are $\mathbf{v}_1^T$ and $\mathbf{v}_2^T$. We compute:

$$\begin{bmatrix} 1 & -4 & 2 \\ -3 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & -11 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Therefore a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in the null space satisfies

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= x_3 \\ x_3 &= x_3 \end{aligned}$$

so we can choose for example $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Solution 2: We want to know when the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Row-reduce the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & x_1 \\ -4 & 1 & x_2 \\ 2 & 5 & x_3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & x_1 \\ 0 & -11 & x_2 + 4x_1 \\ 0 & 11 & x_3 - 2x_1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & x_1 \\ 0 & -11 & x_2 + 4x_1 \\ 0 & 0 & x_3 - 2x_1 + x_2 + 4x_1 \end{array}\right].$$

There will be a solution to the system (which means $\mathbf{x} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$) exactly when the bottom right entry $2x_1 + x_2 + x_3$ is equal to 0. Therefore the equation $2x_1 + x_2 + x_3 = 0$ describes the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

14. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$.

(a) Find a least-squares solution $\hat{\mathbf{x}}$ to the equation $A\mathbf{x} = \mathbf{b}$. Compute $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$.

Solution: Solve $A^T A\mathbf{x} = A^T\mathbf{b}$. We compute

$$A^T A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}.$$

Therefore the solution is

$$\hat{\mathbf{x}} = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 35 \\ 70 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Finally, $\hat{\mathbf{b}} = A\hat{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$.

(b) Construct an orthogonal basis of $\text{Col}(A)$.

Solution: Let $\mathbf{a}_1, \mathbf{a}_2$ be the columns of A. To run Gram-Schmidt, we compute

$$\mathbf{a}_2 - \left(\frac{\mathbf{a}_2 \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}\right) \mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{-1}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/6 \\ 11/6 \\ -5/6 \end{bmatrix}.$$

Therefore an orthogonal basis of $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 11 \\ -5 \end{bmatrix} \right\}$.

(c) Use your answer from part (b) to find the orthogonal projection of $\mathbf{b}$ onto Col(A). Check that your answer is the same as the vector $\hat{\mathbf{b}}$ you computed in (a).

Solution: Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be the orthogonal basis from part (b). The orthogonal projection is

$$\hat{\mathbf{b}} = \left(\frac{\mathbf{b} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{b} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \frac{4}{6}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \frac{70}{210}\begin{bmatrix} 8 \\ 11 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix},$$

which matches the answer from (a).

15. Construct an $m \times n$ matrix A and a vector $\mathbf{b}$ in $\mathbf{R}^m$ such that there is more than one least-squares solution $\hat{\mathbf{x}}$ to the equation $A\mathbf{x} = \mathbf{b}$. You do not need to compute the least-squares solutions, only explain why more than one solution exists.

Solution: If $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto Col(A), any $\hat{\mathbf{x}}$ that satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$. There will be infinitely many such solutions as long as Nul(A) is nontrivial (i.e. has dimension at least 1). So, any $m \times n$ matrix A with nontrivial null space and any $\mathbf{b} \in \mathbf{R}^m$ will satisfy the required property. We could take for example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

16. (a) Find the matrix of the quadratic form $Q(x_1, x_2) = 3x_1^2 - 4x_1x_2 + 6x_2^2$.

Solution: The matrix is $A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$.

(b) Find an orthogonal matrix P such that if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a change of variables, then the new quadratic form (in the variables y_1, y_2) has no cross-term. Write the new quadratic form.

Solution: First find an orthogonal diagonalization of A. To find the eigenvalues, the characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ -2 & 6 - \lambda \end{vmatrix} = (3 - \lambda)(6 - \lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2).$$

The $\lambda = 7$ eigenspace is Nul($A - 7I$):

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

so an eigenvector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

The $\lambda = 2$ eigenspace is orthogonal to the other eigenspace since A is symmetric, so an eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Normalizing the eigenvectors to be unit vectors, we obtain $A = PDP^{-1} = PDP^T$ with

$$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \qquad D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}.$$

Using the change of variables $\mathbf{x} = P\mathbf{y}$, we have $\mathbf{y} = P^T\mathbf{x}$ and $\mathbf{y}^T = \mathbf{x}^T P$, so

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T PDP^T\mathbf{x} = \mathbf{y}^T D\mathbf{y} = 7y_1^2 + 2y_2^2.$$

(c) Classify the quadratic form Q as positive (semi)-definite, negative (semi)-definite, or indefinite. Briefly explain your reasoning.

Solution: The quadratic form is positive definite since both eigenvalues of A are positive. Alternatively, it is clear that $7y_1^2 + 2y_2^2 \geq 0$ with equality only when $\mathbf{y} = \mathbf{0}$.

17. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find an orthonormal set of eigenvectors for the matrix $A^T A$.

Solution: First compute $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Therefore an orthonormal set of eigenvectors is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Note that we list them in this order since $\mathbf{e}_2$ has a larger eigenvalue than $\mathbf{e}_1$.

(b) Find a singular value decomposition $A = U\Sigma V^T$.

Solution: From part (a), we let

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

To find U we compute

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the first column of U will be $\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}$ and the second column of U will be $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$. The third column must complete the orthonormal basis of $\mathbf{R}^3$, so it is in the null space of

$$\begin{bmatrix}1 & 1 & 0\\-1 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & 0\\0 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1/2\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 1/2\end{bmatrix}.$$

The equations are

$$\begin{aligned}x_1 &= \frac{1}{2}x_3\\x_2 &= -\frac{1}{2}x_3\\x_3 &= x_3\end{aligned}$$

so a nonzero vector in the null space is $\begin{bmatrix}1\\-1\\2\end{bmatrix}$, which we normalize to $\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$.

In conclusion, $A = U\Sigma V^T$ with Σ and V as above and

$$U = \begin{bmatrix}-1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}.$$

(c) Determine the rank of A using only your answer from part (b).

Solution: The rank of A is the number of nonzero singular values, which is 2.